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AN EVEN SIMPLER DERIVATION OF THE POWER SPECTRUM OF FULL-RESPONSE CPM

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ABSTRACT

An alternative Markov representation of full-response continuous phase modulation (CPM) leads to a power-spectral expression that is a simple function of two vectors, one representing possible symbol waveforms and one representing possible signaling-interval phase changes. Further, it allows for arbitrary symbol probabilities and arbitrary rational modulation indices.

Keywords: digital communication, continuous phase modulation, CPM

1. INTRODUCTION

The constant envelope of continuous phase modulation (CPM) makes it important in radar and communication systems in which maximum efficiency of the transmitter power amplifier is required. Biglieri and Visintin published a paper in 1990 entitled "A simple derivation of the power spectrum of full-response CPM and some of its properties" [1]. The present result, developed along similar lines but using a different Markov signal representation, is both more compact and more general, and the derivation may even be simpler. The development begins with definitions.

The complex envelope of the general CPM signal is conventionally expressed as

$$\bar{s}(t) = e^{2\pi jh \int_{-\infty}^{t} \sum_k a_k g(t-kT)\, dt},$$

where h is the *modulation index*, where each M-ary symbol a_k can take on any of the values in some symbol alphabet $\{\alpha_1, \ldots, \alpha_M\}$, and where the *frequency pulse* $g(t)$ has support contained in an interval of duration LT and is assumed normalized so that $\int g(t)\, dt = \frac{1}{2}$. Because of this normalization, the net phase change due to symbol a_k is just $\pi h a_k$. When $L = 1$, this entire phase change occurs over an interval of duration T, and the signal is said to be a *full-response* CPM signal, because in a single signaling interval the transmitted signal has completed its response to the symbol to be transmitted. When integer $L > 1$, this is not the case, and *partial-response* CPM results. The most well-known example of a CPM signal is minimum-shift keying (MSK), a full-response CPM signal for which the modulation index $h = \frac{1}{2}$, $g(t)$ is constant for T seconds and is zero elsewhere, each symbol $a_k = \pm 1$, and the net phase change per signaling interval is $\pm\frac{\pi}{2}$.

In what follows $(\cdot)^*$ denotes complex conjugation, $(\cdot)^T$ denotes vector or matrix transposition, and $(\cdot)^\dagger$ denotes both at once, that is, the Hermitian transpose. The convolution operator is the star $\star$, vectors are boldface, and the bar in $\bar{x}$ indicates that $\bar{x}$ is a random process.

The most compact spectral expressions for a general full-response CPM signal have long been those derived using this now-routine result of Prabhu and Rowe [2]:

Theorem (Prabhu and Rowe) *Let $\mathbf{p}(t)$ be a complex vector pulse-shape function, and let $\mathbf{P}(\Omega)$ be its Fourier transform. Let $\{\bar{\mathbf{a}}_n\}$ be a correlation-stationary discrete-time random vector process, and let $S_{\bar{\mathbf{a}}}(\omega)$ be its spectral-density matrix. If time delay u is uniformly distributed between 0 and T and independent of $\{\bar{\mathbf{a}}_n\}$, then the random process*

$$\bar{s}(t) = \sum_k \bar{\mathbf{a}}_k^T \mathbf{p}(t - kT - u)$$

is correlation stationary and has spectral density

$$S_{\bar{s}}(\Omega) = \frac{1}{T}\mathbf{P}^T(\Omega)\, S_{\bar{\mathbf{a}}}(\Omega T)\, \mathbf{P}^*(\Omega).$$

The use of this result to derive CPM spectra has a long history. In a classic 1975 paper, Rowe and Prabhu [3] derived spectra for both full-response and partial-response CPM signals. In 1981 Deshpande and Wittke [4] derived the full-response CPM spectrum for symmetric alphabets as part of a larger paper on optimization of CPM waveforms. Biglieri and Visintin [1] in 1990 published a derivation along present lines for full-response CPM for the special case of integral Mh, using the alternative CPM trellis (Markov formulation) proposed by Rimoldi [5]. The present development removes the Mh restriction and presents the simplest full-response CPM spectral formulation yet.

The main result, presented next and derived in subsequent sections, is a simple spectral expression determined by two vectors, one containing data waveforms and one containing the net phase changes possible in any signaling interval. It allows for arbitrary symbol probabilities and arbitrary rational modulation indices:

This work was done while the author was with Michigan Technological University.

Theorem (Desired Result) *For a unit-amplitude full-response CPM complex envelope, define a complex unit vector*

$$\mathbf{\Psi} = \begin{pmatrix} \sqrt{q_1} e^{j\theta_1/2} \\ \vdots \\ \sqrt{q_M} e^{j\theta_M/2} \end{pmatrix},$$

where $\theta_1, \ldots, \theta_M$ are the M possible values of the net signaling-interval phase change $\pi h a_k$, and where $q_1, \ldots, q_M$ are the associated a priori *probabilities of each. Let frequency pulse $g(t)$ be impulse free with $\int g(t)\, dt = 1/2$, and define*

$$\mathbf{G}(\Omega) = \begin{pmatrix} \sqrt{q_1} P_1(\Omega) \\ \vdots \\ \sqrt{q_M} P_M(\Omega) \end{pmatrix},$$

where $P_i(\Omega)$ is the Fourier transform of the $p_i(t)$ given by

$$p_i(t) = e^{j\theta_i \operatorname{sgn}(t) \star g(t)}$$

for t in an interval of width T containing the support of $g(t)$ and given by $p_i(t) = 0$ for t outside that interval. The power spectrum $S_{\bar{s}}(\Omega)$ of the full-response CPM complex envelope, assuming independent symbols, has a spectral density given by

$$T S_{\bar{s}}(\Omega) = \mathbf{G}^\dagger(\Omega) \operatorname{Re}\left\{ I + \frac{2}{e^{j\Omega T} - \mathbf{\Psi}^T \mathbf{\Psi}} \mathbf{\Psi} \mathbf{\Psi}^T \right\} \mathbf{G}(\Omega)$$

If $|\mathbf{\Psi}^T \mathbf{\Psi}| < 1$ there are no spectral lines, but if $\mathbf{\Psi}^T \mathbf{\Psi} = 1 = e^{j\Omega_\gamma T}$ for some real Ω_γ, then this addtional spectral-line term must be formally added to the right side:

$$\mathbf{G}^\dagger(\Omega) \operatorname{Re}\left\{ \frac{2\pi}{T e^{j\Omega_\gamma T}} \sum_k \delta\left(\Omega - \Omega_\gamma - \frac{2\pi k}{T}\right) \right\} \mathbf{G}(\Omega).$$

The usual CPM signal that would be transmitted on a channel has $|\mathbf{\Psi}^T \mathbf{\Psi}| < 1$ and has the continuous spectral form given first above. The additional spectral-line term typically arises when a conventional CPM signal is passed through a bandpass nonlinearity, as might be used in certain types of synchronization systems for carrier and symbol timing.

The derivation of this result below in Section 3 is based on a representation of the CPM signal, presented next, that is distinct from both the conventional representation and from Rimoldi's alternative representation, but having in common with both that it is a trellis or Markov representation.

2. CPM SIGNAL FORMULATION

Let set of times $I_s \subset \mathbb{R}$ denote the length T zero-th signaling interval, so that signaling interval k is $I_s + kT$, and let ϕ_k represent the signal-phase midpoint for the kth signaling interval. (If the phase trajectory in the interval is sufficiently symmetric, this will be just the signal phase at the temporal midpoint of the interval, but in general no such assumption can be made.)

In general, if the net data-related phase change in signaling interval k is d_k, then

$$\phi_k = \phi_{k-1} + \frac{d_k + d_{k-1}}{2},$$

because for the signal to move from phase ϕ_{k-1} to phase ϕ_k, it must complete the half-done phase shift from interval $k-1$ and then proceed through half the phase shift of the new interval k.

Write the CPM signal as

$$\bar{s}(t) = \sum_k \bar{\mathbf{a}}_k^T \mathbf{p}(t - kT - u)$$

where, using e_i to represent the standard unit vector whose element m is δ_{im}, symbol vector $\bar{\mathbf{a}}_k = e_i e^{j\phi_k}$ when $d_k = \theta_i$, the ith element of phase-change alphabet $\{\theta_1, \ldots, \theta_M\}$, so that the inner product $\bar{\mathbf{a}}_k^T \mathbf{p}(t - kT - u)$ will both select an appropriate signal trajectory from vector $\mathbf{p}(t - kT - u)$ and rotate its phase to set the average of starting and ending phases to the desired value. The phase of element i of $\mathbf{p}(t)$ now needs to go from $-\theta_i/2$ to $\theta_i/2$ in one signaling interval. Therefore, let this ith element $p_i(t)$ be

$$\begin{aligned} p_i(t) &= \exp\left(j\,\theta_i \left[2 \int_{I_s \cap (-\infty, t]} g(t)\, dt - \frac{1}{2} \right] \right) \\ &= e^{\,j\theta_i \operatorname{sgn}(t) \star g(t)} \end{aligned}$$

on I_s and zero elsewhere. Here $g(t)$ is the usual *frequency pulse*, which must be free of impulses and normalized so that its integral over I_s is equal to $\frac{1}{2}$. To verify the second form above, note that its instantaneous radian frequency in the interior of I_s is $\frac{d}{dt}\theta_i \operatorname{sgn}(t) \star g(t) = \theta_i[\frac{d}{dt}\operatorname{sgn}(t)] \star g(t) = \theta_i 2\delta(t) \star g(t) = \theta_i 2 g(t)$, the same as for the first form. The starting and ending phases are easily verified using the normalization for $g(t)$.

3. CPM SPECTRUM DERIVATION

To derive the power spectrum of the correlation-stationary process $\bar{s}(t)$, we first need the spectral-density matrix for the vector symbol stream. Using conditional expectations,

$$\begin{aligned} R_{\bar{\mathbf{a}}}(n) &= E[\bar{\mathbf{a}}_k \bar{\mathbf{a}}_{k-n}^\dagger] \\ &= E\Big[E\Big[E[\bar{\mathbf{a}}_k \bar{\mathbf{a}}_{k-n}^\dagger \mid d_k, d_{k-n}] \mid d_{k-n}\Big]\Big], \end{aligned}$$

for $n > 0$. But if $d_k = \theta_i$ and $d_{k-n} = \theta_l$, then $\bar{\mathbf{a}}_k \bar{\mathbf{a}}_{k-n}^\dagger = e_i e_l^T e^{j(\phi_k - \phi_{k-n})}$. Using

$$\begin{aligned} \phi_k - \phi_{k-n} &= \frac{1}{2} \sum_{m=0}^{n-1} (d_{k-m} + d_{k-m-1}) \\ &= \frac{d_k}{2} + \left(\sum_{m=1}^{n-1} d_{k-m} \right) + \frac{d_{k-n}}{2} \end{aligned}$$

and the independence of $d_{k-n+1}, \ldots, d_{k-1}$, the inner conditional expectation then becomes

$$\begin{aligned} E[\bar{\mathbf{a}}_k \bar{\mathbf{a}}_{k-n}^\dagger \mid d_k = \theta_i, d_{k-n} = \theta_l] \\ &= e_i e_l^T e^{j\theta_i/2} e^{j\theta_l/2} \prod_{m=1}^{n-1} E[e^{j d_{k-m}}] \\ &= e_i e_l^T e^{j\theta_i/2} e^{j\theta_l/2} \gamma^{n-1}, \end{aligned}$$

where $\gamma = E[e^{j d_{k-m}}]$ is assumed the same for each of the several m. Taking the expectation over d_k,

$$E[\bar{\mathbf{a}}_k \bar{\mathbf{a}}_{k-n}^\dagger \mid d_{k-n} = \theta_l] = Q\boldsymbol{\Theta} e_l^T e^{j\theta_l/2} \gamma^{n-1},$$

where

$$\boldsymbol{\Theta} = \begin{pmatrix} e^{j\theta_1/2} \\ \vdots \\ e^{j\theta_M/2} \end{pmatrix}$$

and where Q is a diagonal matrix with Q_{ii} the probability that $d_k = \theta_i$ (assumed independent of k). Likewise, taking the expectation over d_{k-n} yields

$$R_{\bar{\mathbf{a}}}(n) = E[\bar{\mathbf{a}}_k \bar{\mathbf{a}}_{k-n}^\dagger] = Q\boldsymbol{\Theta}\boldsymbol{\Theta}^T Q \gamma^{n-1},$$

for $n > 0$. For $n = 0$, a similar but much simpler line of reasoning leads to

$$R_{\bar{\mathbf{a}}}(0) = E[\bar{\mathbf{a}}_k \bar{\mathbf{a}}_k^\dagger] = E\big[E[e_i e_i^T \mid d_k]\big] = Q.$$

Since $R_{\bar{\mathbf{a}}}(n)$ must have $R_{\bar{\mathbf{a}}}(n) = R_{\bar{\mathbf{a}}}^\dagger(-n)$,

$$R_{\bar{\mathbf{a}}}(n) = \text{H.S.}\{Q\delta_n + 2Q\boldsymbol{\Theta}\boldsymbol{\Theta}^T Q \gamma^{n-1} 1_{[1,\infty)}(n).\} \quad (1)$$

for all n, where the indicator function $1_{[1,\infty)}(n)$, a shifted unit step, is equal to unity when $n \in [1,\infty)$ and zero otherwise. The Hermitian-symmetric component of an arbitrary sequence of matrices A_n is denoted by H.S.$\{A_n\} \triangleq \frac{1}{2}(A_n + A_{-n}^\dagger)$.

To take the Fourier transform of $v_n \triangleq \gamma^{n-1} 1_{[1,\infty)}(n)$, let $h_n = \delta_{n+1} - \gamma\delta_n$, so that $v_n \star h_n = \delta_n$. Then $V(\omega)H(\omega) = 1$, so at frequencies for which $H(\omega) \neq 0$, $V(\omega) = \frac{1}{H(\omega)} = \frac{1}{e^{j\omega} - \gamma}$. For $|\gamma| < 1$, $H(\omega) \neq 0$ for all ω, so this is the desired result. If $|\gamma| = 1$ however, $\gamma = e^{j\omega_\gamma}$ for some frequency ω_γ, and it follows that $H(\omega_\gamma) = 0$. Any component then of v_n at frequency ω_γ must be discovered some other way. But the amplitude of any component of v_n proportional to $e^{j\omega_\gamma n}$ is given by the time average of $v_n e^{-j\omega_\gamma n}$. Here, if $V(\omega)$ has a component $\alpha \sum_k \delta(\omega - \omega_\gamma - 2\pi k)$, then

$$\alpha = \lim_{N\to\infty} \frac{2\pi}{2N+1} \sum_{n=-N}^{N} v_n e^{-j\omega_\gamma n} = \pi e^{-j\omega_\gamma}.$$

So,

$$V(\omega) = \frac{1}{e^{j\omega} - \gamma} + \begin{cases} \pi e^{-j\omega_\gamma} \sum_k \delta(\omega - \omega_\gamma - 2\pi k) \\ \text{if } \gamma = e^{j\omega_\gamma} \end{cases}$$

The Fourier transform of (1) can now be written[1] $S_{\bar{\mathbf{a}}}(\omega) = \text{Re}\left\{Q + 2Q\boldsymbol{\Theta}\boldsymbol{\Theta}^T Q V(\omega)\right\}$. Prabhu and Rowe's theorem yields the CPM power spectrum

$$TS_{\bar{s}}(\Omega) = \mathbf{P}^T(\Omega)\,\text{Re}\left\{Q + 2Q\boldsymbol{\Theta}\boldsymbol{\Theta}^T Q V(\Omega T)\right\}\mathbf{P}^*(\Omega)$$

For brevity, define

$$\begin{aligned} \mathbf{G}(\Omega) &= Q^{\frac{1}{2}}\mathbf{P}(\Omega) \\ \boldsymbol{\Psi} &= Q^{\frac{1}{2}}\boldsymbol{\Theta}. \end{aligned}$$

Then $\gamma = \boldsymbol{\Theta}^T Q \boldsymbol{\Theta} = \boldsymbol{\Psi}^T\boldsymbol{\Psi}$, and the CPM spectrum becomes

$$TS_{\bar{s}}(\Omega) = \mathbf{G}^T(\Omega)\,\text{Re}\left\{I + \boldsymbol{\Psi}\boldsymbol{\Psi}^T \frac{2}{e^{j\Omega T} - \boldsymbol{\Psi}^T\boldsymbol{\Psi}}\right\}\mathbf{G}^*(\Omega)$$

if $|\boldsymbol{\Psi}^T\boldsymbol{\Psi}| < 1$ and

$$\begin{aligned} TS_{\bar{s}}(\Omega) = \\ &\mathbf{G}^T(\Omega) \\ &\times \text{Re}\left\{I + \boldsymbol{\Psi}\boldsymbol{\Psi}^T\left(\frac{2}{e^{j\Omega T} - \boldsymbol{\Psi}^T\boldsymbol{\Psi}} + \frac{2\pi}{T e^{j\Omega_\gamma T}}\sum_k \delta\left(\Omega - \Omega_\gamma - \frac{2\pi k}{T}\right)\right)\right\} \\ &\times \mathbf{G}^*(\Omega), \end{aligned}$$

if $\boldsymbol{\Psi}^T\boldsymbol{\Psi} = e^{j\Omega_\gamma T}$, where $\omega_\gamma = \Omega_\gamma T$ has been used for convenience.

Conjugating the right side of these real spectral expressions is harmless, so the desired result follows.

4. SUMMARY

In the common Markov representation of a CPM signal the Markov state stores the phase of the signal at the most recent symbol-interval boundary. A set of data waveforms, one for each possible data symbol, encodes the possible intra-signaling-interval excess-phase trajectories relative to this state. Here in contrast the state stores the average of the signal phases at the beginning and end of the signaling interval. The data waveforms encode the excess phase relative to this average phase. This change results in a CPM signal formulation that is in some sense more symmetric than the usual, resulting here in an elegant expression for the CPM spectrum that is also considerably more general than previous results.

5. REFERENCES

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[1] In general, the Fourier transform of H.S.$\{A_n\}$ is the Hermitian component of $\mathcal{A}(\omega)$, which is given by $\frac{1}{2}[\mathcal{A}(\omega) + \mathcal{A}^\dagger(\omega)]$, where $\mathcal{A}(\omega)$ is the Fourier transform of A_n. But here the matrix $\mathcal{A}(\omega)$ is symmetric, so its Hermitian component is just its real part.

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