

Superdirective and SNR Constraints in Wideband Array-Pattern Design

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Abstract—Wide instantaneous- and tunable-bandwidth arrays pose a challenging design problem. Since the elements are usually spaced close to one-half wavelength at the highest frequency of operation, at lower frequencies the array may be considerably oversampled spatially. In a conventional time-delay steered array the result is a beamwidth that is proportional to frequency. With an FIR filter at each element, however, superdirective can be achieved at lower frequencies, improving the beam characteristics. Constraints on efficiency, sensitivity, and SNR are derived to limit the undesirable effects of superdirective.

I. INTRODUCTION

It has become increasingly practical to implement wideband antenna arrays digitally, with an A/D and/or D/A converter at each element and FIR filtering replacing the traditional phase shifter and time delay. Correspondingly, we are no longer bound to cookbook design methods and can use optimization to custom-design the array pattern for a desired application [1–3]. In this paper we consider optimizing the directivity of a wideband array, subject to constraints on transmit array efficiency and sensitivity and receive array signal-to-noise ratio (SNR).

The directivity of an antenna is usually defined as the ratio of its maximum radiation intensity to the radiation intensity of an isotropic element transmitting the same power [4]. In antennas with a single main beam and no high sidelobes, directivity is largely a measure of beamwidth. For a half-wavelength spaced narrowband linear array of isotropic elements, uniform illumination maximizes directivity. In arrays with closer element spacing higher directivities are possible than with uniform illumination, an effect known as *superdirective*. Superdirective in narrowband arrays was a major research topic in the 1940’s, although it may have been discovered earlier ([5] provides some discussion and a bibliography). Schelkunoff [6] noted the effect in endfire arrays. Bouwkamp and deBruijn [7] showed that, in theory, a continuous line antenna could have arbitrarily high directivity by proper choice of the current. Other authors [8–10] examining superdirective arrays determined that a) the maximum directivity of a closely spaced array is finite but larger than that of a uniformly illuminated array, and that b) decreasing the spacing for a fixed physical array length by adding more elements allows arbitrarily high directivities.

In wideband arrays, directivity becomes a function of frequency. Because a given antenna is smaller (measured in wavelengths) at lower frequencies, a uniformly illuminated and

time-delay steered wideband pattern broadens from the upper band edge to the lower band edge, with a corresponding reduction in directivity. Since the element spacing is usually chosen based on the highest operating frequency to prevent grating lobes, at lower frequencies the element spacing is less than one-half wavelength. A pattern that is superdirective at the lower frequencies can have a more constant beamwidth across frequency. However, superdirective also has negative performance implications for both transmit and receive arrays. In transmit arrays superdirective leads to greater currents and resistive losses in the elements and thus lowered efficiency. In receive arrays, superdirective leads directly to increased noise gain and decreased SNR. In both cases the array pattern becomes more sensitive to mismatch between the elements.

Recently, numeric optimization has been used to design superdirective narrowband arrays [11–13]. This paper seeks to extend previous results to wideband antenna arrays, and to formulate superdirective constraints in a form suitable for solving as a second-order cone (SOC) program using any of an increasing number of highly efficient solvers [14, 15]. By formulating and constraining efficiency, sensitivity, and noise gain directly in the pattern optimization, we can achieve improved performance relative to conventional time-delay methods along with tremendous flexibility of design.

II. WIDEBAND ARRAY PARAMETERS

A. Array Pattern

We initially assume the array structure of [1–3]: a general array of isotropic elements, each connected to an A/D and/or D/A converter and an FIR filter. The far-field array pattern

$$A(\boldsymbol{\nu}, f) = \sum_{\mathbf{x} \in \mathcal{X}} H_{\mathbf{x}}(f) e^{-j2\pi\boldsymbol{\nu} \cdot \mathbf{x}} \quad (1)$$

is the spatial and temporal frequency response of the array, where \mathcal{X} is the set of element locations, and

$$H_{\mathbf{x}}(f) = \sum_n h_{\mathbf{x}, n} e^{-j2\pi n f T} \quad (2)$$

is the frequency response of the FIR filter at location \mathbf{x} , operating at a rate $1/T$. The spatial frequencies $\boldsymbol{\nu}$ corresponding to physically realizable, propagating plane waves lie on the cone defined by the Helmholtz equation $\|\boldsymbol{\nu}\| = |f|/c$. For a given temporal frequency f , these Helmholtz spatial frequencies occupy a spherical shell representing the full field of view in three dimensions. Unit vector \mathbf{u} , representing the look direction of

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either a transmit or receive array, corresponds to a spatial frequency of $\nu = -\mathbf{u}f/c$. Off the Helmholtz cone the array pattern value doesn't correspond to propagating radiation but it is important as it relates to other array performance measures.

For example, consider a simple linear array on the z -axis with uniform element spacing d_z . Now the pattern is constant in spatial-frequency components ν_x and ν_y , and periodic in ν_z and f with periods $1/d_z$ and $1/T$ respectively. Specifying the pattern over any one period in the (ν_z, f) plane thus defines the entire pattern. Fig. 1 shows that the region defined by the Helmholtz cone over the frequency interval $[f_{lo}, f_{hi}]$ forms a trapezoid with its narrow edge at low temporal frequencies. The shaded rectangle represents one spacial-temporal frequency period of the array pattern when $d_z = c/(2f_{hi})$. Regions inside the rectangle but outside the central trapezoid are “don't care” in an optimization of just the pattern. They represent the FIR filter transition bands (the top and bottom strips) as well as the non-propagating frequencies (the triangles). Superdirective results from leaving these regions unconstrained, so that a narrower beam is realized than if the pattern was constrained outside the Helmholtz cone. However, the pattern may grow unacceptably large in those regions. Limiting that effect is the focus of the next section.

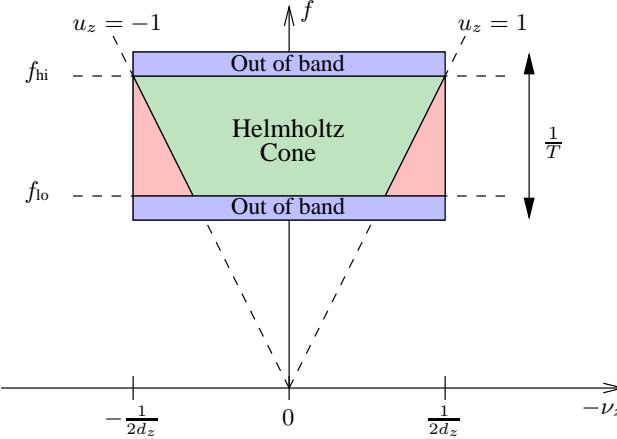


Figure 1: Array pattern regions versus frequency.

B. Wideband Directivity

The directivity of a narrowband array can be defined in terms of the antenna power pattern as the ratio of the maximum power pattern to the average power pattern over visible space:

$$D_0(f) \triangleq \frac{|A_0(f)|^2}{P_{\text{avg}}(f)}, \quad (3)$$

where $A_0(f) \triangleq A(-\mathbf{u}f/c, f)$ is the maximum value of the antenna pattern and $P_{\text{avg}}(f)$ is the average of the power pattern over the sphere of look directions at frequency f ,

$$P_{\text{avg}}(f) \triangleq \frac{1}{4\pi} \oint_{\|\mathbf{u}\|=1} |A(-\mathbf{u}f/c, f)|^2 d\Omega. \quad (4)$$

We will assume that \mathbf{u}_0 , the direction of the pattern maximum, is the direction to which the beam has been steered and is constant in frequency. Clearly directivity is not directly affected by the value of the pattern outside the Helmholtz cone. We will define *wideband* directivity as

$$D_{0\text{wb}} \triangleq \frac{\frac{1}{B} \int_{\mathcal{F}_{\text{pb}}} |A_0(f)|^2 df}{\frac{1}{B} \int_{\mathcal{F}_{\text{pb}}} P_{\text{avg}}(f) df}, \quad (5)$$

averaging the numerator and denominator across the passband. Wideband directivity provides a single figure of merit to be maximized. In a typical optimization the maximum pattern $A_0(f)$ will be approximately fixed, in which case minimizing the quadratic denominator of (5) effectively maximizes the directivity.

As will be seen in the examples, simply maximizing directivity without some other constraint will often lead to impractical designs. Since directivity is only a function of the pattern on the Helmholtz cone, elsewhere the pattern may be arbitrarily large. In the next section we consider how this affects the performance of transmit and receive arrays, and formulate appropriate constraints.

III. WIDEBAND ARRAY CONSTRAINTS

In this section we will present wideband extensions to several narrowband constraints that have been used in the past to limit the undesired behavior of superdirective transmit and receive arrays. The goal is to add a new set of potential optimization constraints to those derived in [1–3].

A. Transmit Constraints

Often the primary concern in a transmit array is placing the most radiated energy on a target, and thus the interest in maximizing directivity. However, increasing the directivity beyond that of a uniformly illuminated array while holding the peak of the pattern fixed actually results in higher currents in the antenna elements, without a corresponding rise in transmitted power. This in turn leads to greater resistive losses in the elements, lowering the efficiency of the array. As the currents rise the array pattern becomes increasingly sensitive to small differences between the elements. Finally, when a D/A converter is used at each element, we need to keep the signals within its dynamic range. Each of these can serve as a constraint on directivity to ensure the practicality of the resulting array.

Since the input to each FIR filter is assumed identical, the apparent power density driving each element is proportional to the power gain $|H_x(f)|^2$ of its corresponding filter. The total input power gain across frequency is then

$$G(f) \triangleq \sum_{x \in \mathcal{X}} |H_x(f)|^2, \quad (6)$$

and the average power gain across the band is

$$G_{\text{wb}} \triangleq \frac{1}{B} \int_{\mathcal{F}_{\text{pb}}} G(f) df. \quad (7)$$

It is straightforward to show that for an N -element array with $A_0(f)$ fixed, the minimum input power gain $G(f)$ is

$$G_{\min}(f) = \frac{|A_0(f)|^2}{N},$$

which is obtained when

$$H_x(f) = \frac{A_0(f)}{N} e^{-j2\pi\mathbf{x}\cdot\mathbf{u}_0 f/c}.$$

This also minimizes the average power gain G_{wb} . In the common case where $A_0(f) = A_0$ is constant, this represents uniform illumination with time-delay filters. Any increase in directivity above the uniform case will necessarily increase G_{wb} , and thus the power into the array. We can accordingly define *power efficiency* as the ratio of the minimum input power gain to the actual input power gain, both as a function of frequency,

$$\eta_2(f) \triangleq \frac{|A_0(f)|^2}{NG(f)},$$

and across the passband

$$\eta_{2\text{wb}} \triangleq \frac{\frac{1}{B} \int_{\mathcal{F}_{\text{pb}}} |A_0(f)|^2 df}{NG_{\text{wb}}}.$$

Essentially equivalent metrics for narrowband arrays can be found in [13, 16].

Dawoud and Anderson [17] introduced a slightly different quantity they called *radiation efficiency*, which also ranges from zero to one:

$$\eta_1(f) = \frac{|A_0(f)|}{\sum_{\mathbf{x} \in \mathcal{X}} |H_x(f)|}. \quad (8)$$

The wideband extension is then

$$\eta_{1\text{wb}} = \frac{\frac{1}{B} \int_{\mathcal{F}_{\text{pb}}} |A_0(f), f| df}{\frac{1}{B} \int_{\mathcal{F}_{\text{pb}}} \sum_{\mathbf{x} \in \mathcal{X}} |H_x(f)| df} \quad (9)$$

Power efficiency and radiation efficiency differ in the norm used, but in general $\eta_2(f) \approx \eta_1^2(f)$.

A related effect of increased filter gain is that the output range of each D/A converter must be increased as well. Simply increasing a later amplifier gain would result in a loss of resolution. To ensure that each D/A's data remains in some desired range, separate constraints can be placed on some measure of each filter's gain in the passband. Depending on the actual waveforms used, the mean-square or maximum gain might be constrained.

A third drawback of increased filter gain is an increased sensitivity of the pattern to element mismatch and position error. Uzsoky and Solymar [18] and Gilbert and Morgan [19] studied the effects of uncorrelated errors in narrowband arrays, and calculated the average intensity of the *error pattern*. For element gain errors of variance ϵ^2 and element position errors of

variance σ^2 the average error pattern intensity at frequency f is $\Delta^2(f)G(f)$, where

$$\Delta^2(f) = \epsilon^2 + \frac{(2\pi f)^2}{3c^2} \sigma^2.$$

Averaging over the passband yields the corresponding wideband error intensity. The ratio of the error pattern intensity to the desired power pattern maximum is proportional to the *sensitivity factor* [19]

$$K(f) = \frac{G(f)}{|A_0(f)|^2}, \quad (10)$$

which is just $1/(N\eta_2(f))$. Because of the dependence of $\Delta^2(f)$ on frequency there is no obvious wideband equivalent of the sensitivity factor unless one source of error dominates. If element gain errors dominate, then we can define wideband sensitivity as $1/(N\eta_{2\text{wb}})$. If position errors dominate then

$$K_{\text{wb}} \approx \frac{\frac{1}{B} \int_{\mathcal{F}_{\text{pb}}} \frac{(2\pi f)^2}{3c^2} G(f) df}{\frac{1}{B} \int_{\mathcal{F}_{\text{pb}}} |A_0(f)|^2 df}.$$

Constraints on efficiency, DAC range, and sensitivity all essentially involve limiting the magnitude of the FIR filter responses, either individually or combined. In the next section similar constraints will be derived for receive arrays.

B. Receive Constraints

In the receive array the bulk of the processing follows the A/D, so dynamic range and efficiency are not problems for the beamformer. Sensitivity for the receive array is defined exactly as for transmit arrays. Most importantly, SNR provides a natural constraint on directivity. In general this will differ from transmit array constraints in that SNR depends both on the array hardware and the environment. Under common assumptions on noise distributions, however, SNR constraints are identical to those derived for transmit arrays.

Two major sources of noise in a receive array are background radiation entering the antenna itself and thermal noise from the receivers. The background radiation is amplified only by the array pattern on the Helmholtz cone, but the receiver noise is amplified by the individual filters. Thus superdirectivity increases the noise gain by increasing the filter response magnitudes. We define two input noise spectral densities: the background antenna noise density $S_a(\mathbf{u}, f)$, which is direction-dependent, and the receiver noise density $S_r(f)$, which will be assumed to be identical for each receiver. The corresponding output spectral densities after beamforming are

$$S_{\text{ao}}(f) = \oint_{\|\mathbf{u}\|=1} S_a(\mathbf{u}, f) |A(-\mathbf{u}f/c, f)|^2 d\Omega \quad (11)$$

and

$$S_{\text{ro}}(f) = S_r(f) G(f), \quad (12)$$

Ex.	$D_{0\text{wb}}$	$G_{\text{wb}} \approx K_{\text{wb}}$	$\eta_{2\text{wb}}$	$\eta_{1\text{wb}}$
1	11.4	35.2	-47.0	-39.2
2	10.8	-11.7	0.0	0.0
3	11.2	-1.8	-10.0	-8.3

Table 1: Summary of example results. All values are in dB.

assuming that the noise from each receiver is uncorrelated with the others. The average noise power in the band of interest is then

$$P_n = \int_{\mathcal{F}_{\text{pb}}} S_{\text{ao}}(f) df + \int_{\mathcal{F}_{\text{pb}}} S_{\text{ro}}(f) df. \quad (13)$$

The power spectrum of the desired signal in the peak of the main beam is multiplied by $|A_0(f)|^2$, with signal power gain proportional to

$$P_s \sim \frac{1}{B} \int_{\mathcal{F}_{\text{pb}}} |A_0(f)|^2 df. \quad (14)$$

SNR is then proportional to P_s/P_n . Again, we will typically fix P_s in a design and optimize or constrain P_n . If we assume no receiver noise and a white background noise density, we see that SNR differs from the wideband directivity defined previously by a constant scale factor. Thus it is the acceptable receiver noise power that actually limits the amount of directivity in a receive array. We can see that when (white) background noise dominates, SNR is proportional to directivity, and when (white) receiver noise dominates, SNR is proportional to power efficiency as defined for transmit arrays. From the derivation of efficiency we see that white receiver noise power is minimized by a uniformly weighted, time-delay steered array.

IV. EXAMPLES

We now present three example optimizations to demonstrate the relatively modest improvement possible with a partially superdirective wideband array. The results are summarized in Table 1 and Fig. 2, in terms of the directivity, sensitivity, receiver noise gain, and efficiency. The example array has $N = 15$ elements uniformly spaced on the z -axis at locations kd for $k = -7, \dots, 7$. With isotropic elements the resulting antenna pattern is a function only of $u_z = \sin(\theta)$ in terms of direction, or ν_x in terms of spatial frequency. Each element has an associated complex-coefficient FIR filter of length 11 operating at a rate of $f_s = 1/T_s$. The array has conjugate symmetry about its center point: $h_{-\mathbf{x}, -n} = h_{\mathbf{x}, n}^*$, which results in a real-valued array pattern. The array center frequency is $1.5f_s$, with a bandwidth $B = 0.8f_s$, so that $\mathcal{F}_{\text{pb}} = [1.1f_s, 1.9f_s]$. The element spacing d is chosen to be one-half wavelength at the upper band edge of $1.9f_s$. At the lower band edge d is approximately 0.29 of a wavelength. The beam will be steered to $u_{z0} = \sin(45^\circ)$, as larger improvements in directivity are possible at greater off-boresight angles.

To see the limits of performance, we first optimize directivity, and thus SNR in a receiver-noise-free system, by minimizing the average of the antenna power pattern while constraining

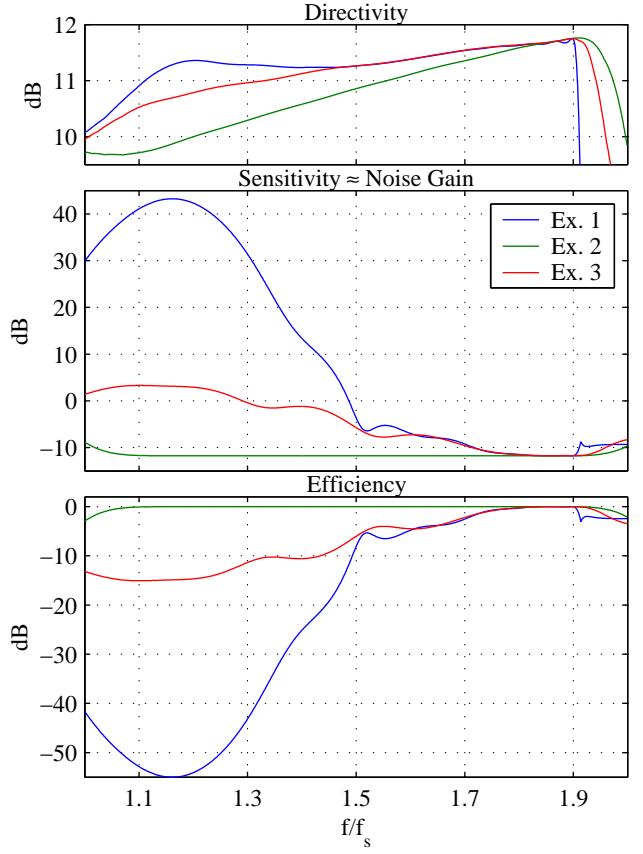


Figure 2: Results of example optimizations: pattern, directivity, sensitivity, and efficiency.

the main beam gain:

$$\text{minimize } \delta \geq 0$$

$$\text{subject to } \frac{1}{B} \int_{\mathcal{F}_{\text{pb}}} A_0(f) df = 1 \quad (15)$$

$$\frac{1}{B} \int_{\mathcal{F}_{\text{pb}}} |A_0(f) - 1|^2 df \leq 10^{-5} \quad (16)$$

$$\frac{1}{B} \int_{\mathcal{F}_{\text{pb}}} P_{\text{avg}}(f) df \leq \delta^2 \quad (17)$$

Nonnegative auxiliary variable δ provides the linear objective required by many solvers. Linear equality constraint (15) and quadratic inequality constraint (16) set the average gain at u_{z0} to unity and the mean-squared error ripple to less than -50 dB across the passband. Fixing the main beam in this way results in $\eta_2(f) \approx 1/(NG(f))$, $\eta_{2\text{wb}} \approx 1/(NG_{\text{wb}})$, $K(f) \approx G(f)$, and if element location errors are negligible, $K_{\text{wb}} \approx G_{\text{wb}}$. Quadratic constraint (17) then serves to minimize the average power pattern and thus maximize directivity. The optimized main beam has nearly constant angular width at all in-band frequencies, as shown in Fig. 3. Fig. 4, which shows the FIR filter frequency responses, and Fig. 5, which shows the optimized pattern versus spatial frequency, illustrate the side-effects of

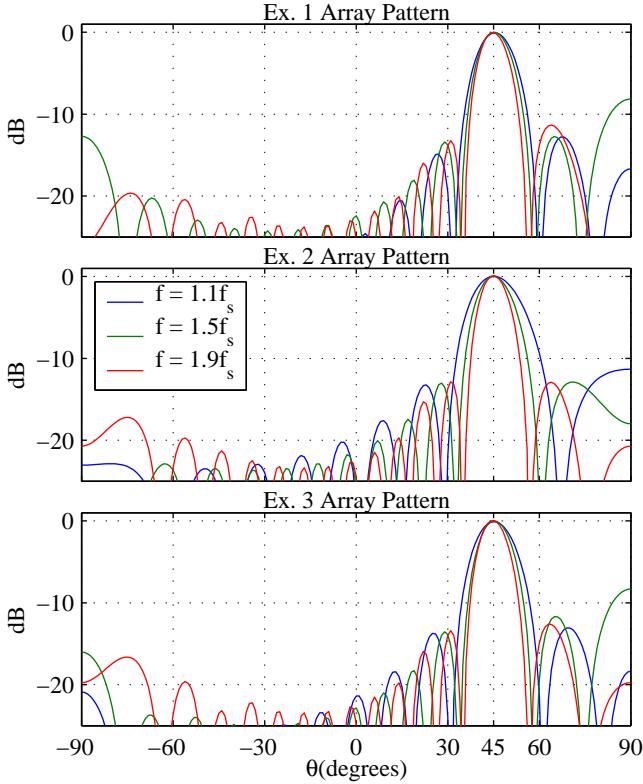


Figure 3: Example patterns vs. angle and frequency.

this improvement. The frequency responses are increasingly large towards the lower frequencies, and the pattern outside the propagating region is over 50 dB greater than the main-beam. The cost is extremely low efficiency and high sensitivity for a transmit array, and an exceedingly large receiver noise gain in a receive array. Without any constraints limiting SNR, sensitivity, efficiency, etc., the result is entirely impractical.

For a second example, we consider a completely non-superdirective design—an FIR approximation of a uniformly illuminated time-delay-steered array. This is easily accomplished by adding the constraint $G_{\text{wb}} \leq 1/N$ to the previous design. By forcing the sensitivity and receiver noise gains to their minimum values and maximizing efficiency, the filters are forced to approximate time delays. Fig. 4 verifies that the filter frequency responses are approximately flat and equal. The negative result is a loss of overall directivity and a wider beam in angle, primarily in the lower frequencies. Fig. 3 shows the characteristic beam broadening of a time-delay array at the lower band edge. Fig. 5 shows that the pattern has low sidelobes even for non-propagating spatial frequencies.

For the third example, we choose a limited superdirective design, constraining the efficiency, sensitivity and noise gain to be within 10 dB of the time-delay limit by adding the constraint $G_{\text{wb}} \leq 10/N$ to the first design. The resulting design has a higher and more constant directivity across frequency than the time-delay design, while retaining manageable sen-

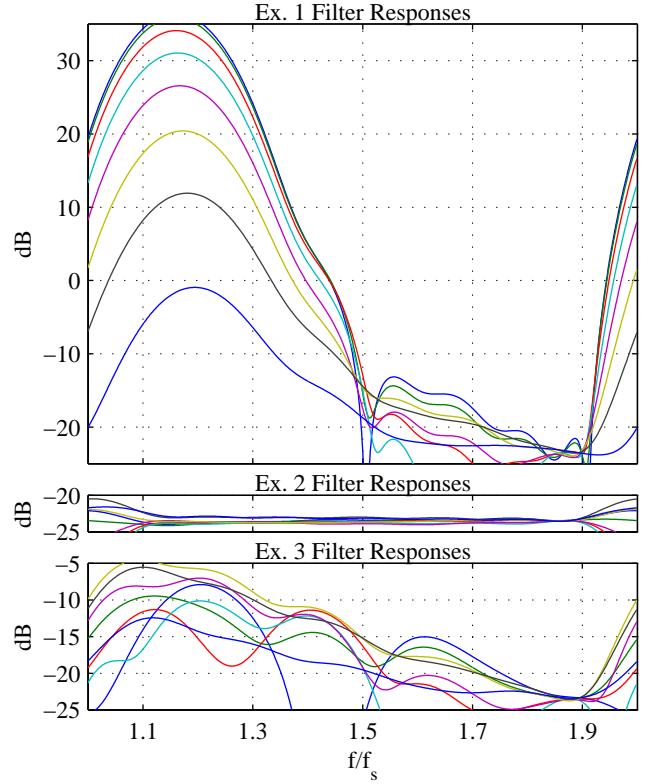


Figure 4: Example filter frequency responses.

sitivity, noise gain, and efficiency.

V. CONCLUSIONS

The designs here were deliberately kept as simple as possible for illustration's sake. Incorporating nonuniform spectral densities, peak sidelobe constraints, filter magnitude constraints, etc., is outlined in [1, 2]. It is also straightforward to consider the near-field pattern, as has been done for narrowband arrays [20, 21]. An important extension is to consider physical array elements and mutual coupling effects, which will likely be significant in a wideband array.

The primary usefulness of superdirectivity in wideband arrays is that it mitigates some of the beam broadening inherent in a time-delay-based array, allowing for a more constant beamwidth and directivity across frequency. As many prior authors have noted, significant increases in directivity lead to unrealizable designs, and so constraints of the kind presented here are needed to ensure a practical design.

Acknowledgment

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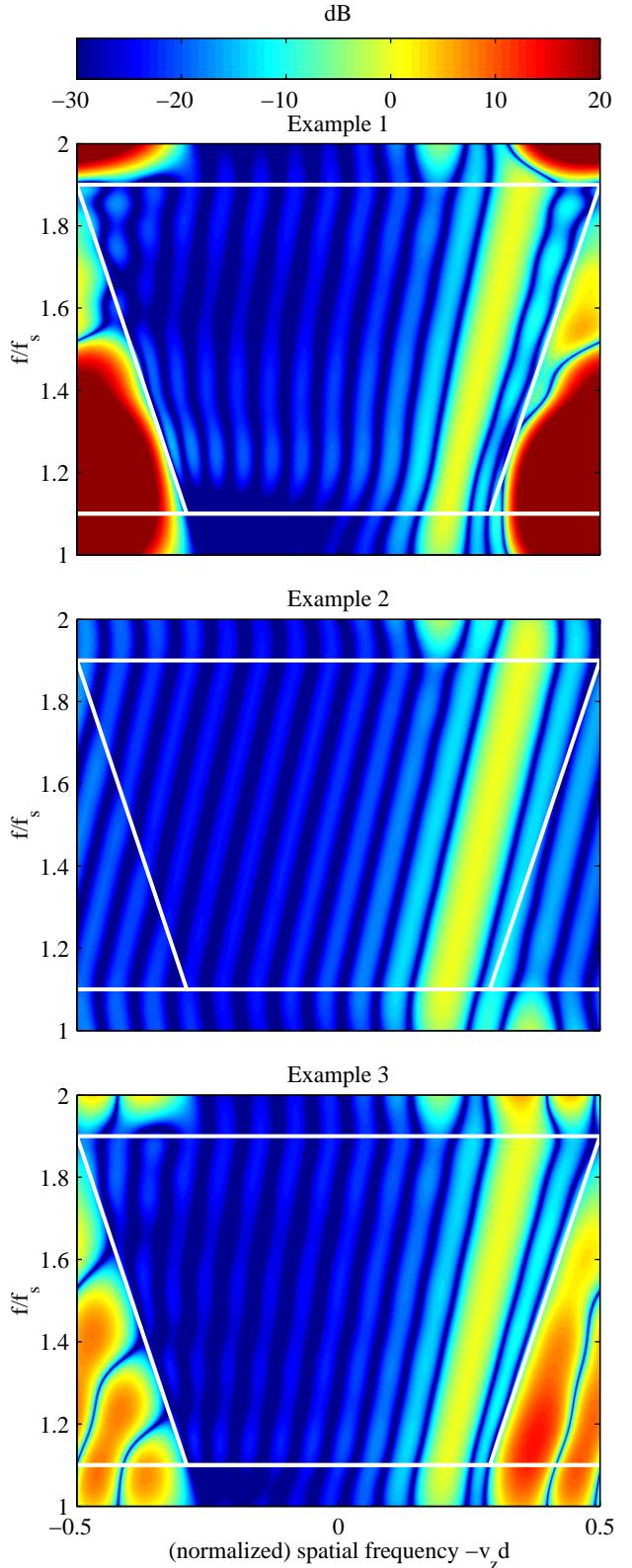


Figure 5: Example patterns vs. spatial and temporal frequency.