Choosing Nonuniform Tap Spacings for a Tapped-Delay-Line Filter

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Abstract— **The family of frequency responses that can be generated by adjusting the weights on a tapped-delay-line filter with nonuniformly spaced taps is just the span of a basis set whose elements are *arbitrary* (as long as linear independence is preserved) frequency shifts of the *characterizing function* of the set of delays. The characterizing function is just the frequency response that results when uniform weights are applied at those delays. If the delays are chosen to provide a unimodal characterizing function with low sidelobes (probability theory provides one approach), then it tends to be easy to approximate a desired frequency response from linear combinations of various frequency shifts of the characterizing function. If the characterizing function is severely multimodal, it is relatively difficult to get the linear combination of the characterizing-function shifts to approximate a desired response. Though this intuition about what is possible comes from visualizing the manual construction of the desired response, the conclusions reached generally also predict the degree of success that will be obtained from an optimal design of the filter weights. Further and most importantly, the characterizing-function idea offers guidance in the choice of the delays themselves.**

Keywords— **FIR filters, arrays, radar, MTI filters.**

I. Introduction

A conventional FIR filter has a frequency response that can be expressed as the weighted sum of a family of basis functions, each of which is a frequency-shifted Dirichlet kernel (Fig. 1). Conventionally the main lobes of the functions are spaced uniformly across the frequency interval. It is relatively easy to visualize the sorts of frequency responses that can be created as weighted sums of such basis elements, because the elements themselves are well-behaved functions, and their dominant peaks are spaced evenly across the frequency band. Most filter designers in fact have a well-developed intuitive sense about what frequency responses are feasible to build by adjusting the weights on a given tapped delay line, as long as those taps are uniformly spaced in delay. This taken-for-granted intuition (rather than various celebrated formulae) generally gives the designer a crude feel for the filter order required.

When the taps are unevenly spaced intuition often fails, leaving the designer with no clue as to where to begin. Further, there are several common sorts of situations in which uneven spacing is the norm. For example, filters that are essentially tapped-delay-line filters with nonuniformly spaced taps arise routinely in radar signal processing, where they are used in conjunction with staggered pulse repetition intervals to filter out low-Doppler (more-or-less physically stationary) clutter. Nonuniformly spaced antenna and sensor arrays are likewise routine, and of course the mathematics of far-field array patterns is identical to that of tapped-delay-line frequency responses. FIR filters with certain taps constrained to be zero, as in nth-band filters, can technically be considered to display uneven tap spacing as well, even though the tap locations are a subset of a uniformly spaced lattice.

So, what sort of requirements can reasonably be placed on a nonuniform-tap-spacing filter? What choices of passband and stopband edges, for example, will offer hope of reasonable results? How can we intuitively sense what a particular placement in delay of a set of taps is "good for"? Part II develops the notion of the characterizing function of a set of delays, a notion that provides a source of intuition about questions like these. Part A shows how a probability-based approach used routinely in the design of large arrays of equally weighted antenna elements can also be used here to choose delays that will enable high-performance filter design.

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II. The Characterizing Function

How can we characterize the family of frequency responses that can be realized by adjusting the weights $h_0, \ldots, h_{N-1}$ associated with a set of arbitrary delays $\tau_0, \ldots, \tau_{N-1}$ on a delay line? Once the weights are known, the frequency response will be given by

$$H(f) = \sum_{n=0}^{N-1} h_n e^{-2\pi i f \tau_n}. \tag{1}$$

Suppose frequencies $\nu_0, \ldots, \nu_{N-1}$ are such that the matrix $\mathbf{A}$ given by $\mathbf{A}_{nk} = e^{2\pi i \nu_k \tau_n}$ is of full rank. Then there are constants $H_0, \ldots, H_{N-1}$ such that for $n = 0, \ldots, N-1$,

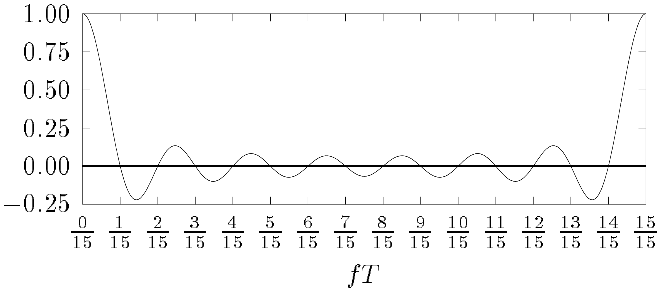
$$h_n = \frac{1}{N} \sum_{k=0}^{N-1} H_k e^{2\pi i \nu_k \tau_n}, \tag{2}$$

which when substituted into (1) yields

$$H(f) = \sum_{k=0}^{N-1} H_k \underbrace{\frac{1}{N} \sum_{n=0}^{N-1} e^{-2\pi i (f-\nu_k)\tau_n}}_{\triangleq\, \mathcal{D}(f-\nu_k)}. \tag{3}$$

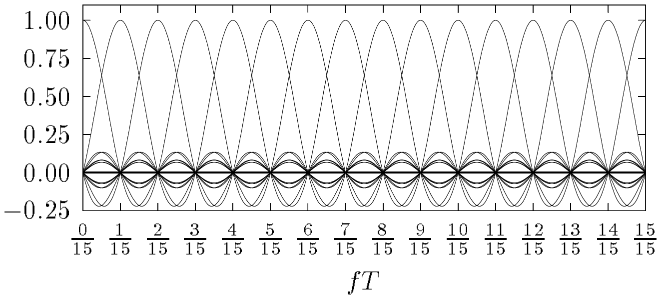
Therefore, $H(f)$ can be expressed as a linear combination of frequency-shifted versions of a *characterizing function*

$\mathcal{D}(f)$:



(a)

$\mathcal{D}\left(f - \frac{k}{15T}\right)$ for $k = 0, \ldots, 14$:



(b)

Fig. 1. The (a) Dirichlet-kernel characterizing function and (b) related conventional basis set for 15 uniformly spaced taps.

$\mathcal{D}(f)$ whose shape is determined by the $\{\tau_n\}$. Conveniently, this function $\mathcal{D}(f)$ is just the frequency response of a tapped-delay-line filter with uniform weights $h_n = 1/N$ at all delays $\tau_0, \ldots, \tau_{N-1}$.

A. Uniform Tap Spacing

Uniformly spaced taps provide an important and familiar special case. Suppose that N is odd, and write $N = 2M + 1$ for convenience. Further suppose that the delays $\tau_0, \ldots, \tau_{N-1}$ are given by $-MT, \ldots, -T, 0, T, \ldots, MT$. Then

$$\mathcal{D}(f) = \frac{1}{N}\sum_{n=-M}^{M} e^{-2\pi i f T n} = \frac{2}{N} D_M(2\pi f T),$$

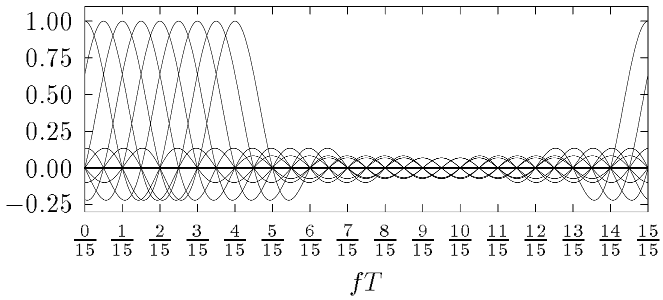
where $D_M(x) \triangleq \frac{1}{2} + \sum_{n=1}^{M} \cos nx$ is the *Dirichlet kernel*[1] used by Zygmund in his analysis classic, *Trigonometric Series* [1]. Figure 1(a) shows one period of this $\mathcal{D}(f)$ for $N = 15$.

Nothing has been said thus far about the choice of frequencies $\{\nu_k\}$, because, as (3) shows, the shape of $\mathcal{D}(f)$ is not affected by the choice. Any $\{\nu_k\}$ that result in linear independence of the $\{\mathcal{D}(f - \nu_k)\}$ can be used, which for uniformly spaced taps means any $\{\nu_k\}$ such that no two of the $\{\nu_k\}$ are separated by an integral multiple of the period T^{-1}. The conventional choice for uniformly spaced taps is $\nu_k = \frac{k}{NT}$, so that (2) becomes an inverse discrete Fourier transform. In the $N = 15$ example, $H(f)$ is a linear combination of the functions in Fig. 1(b). A glance at this family of functions gives an instant feel for how much control of the frequency response is possible with 15 uniformly spaced taps.

Choosing the Frequency Shifts

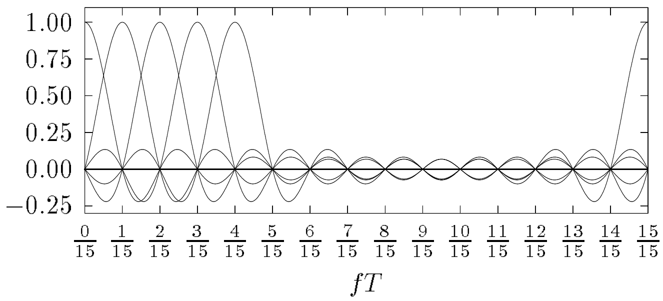
Spacing the frequency shifts $\{\nu_k\}$ of the characterizing functions uniformly across the frequency band does not necessarily provide the most insight into the frequency-response generating capability of a given set of delays, however. Sometimes it is better to visualize building a desired frequency response by choosing the frequency shifts and the weights simultaneously. For example, suppose we choose nine narrowly spaced shifts that will cause the nine characterizing function peaks to be spaced across an intended passband:

$\mathcal{D}(f - \nu_k)$ for $k = 0, \ldots, 8$:



The other six shifted characterizing functions in the fifteen-element basis set can have shifts and weights chosen to try to cancel the sidelobe structure that will result when the passband-related basis elements above are summed.[2] However, if we use only five basis elements, say, to form the passband, the more-widely spaced shifts will be expected to lead to a larger passband ripple:

$\mathcal{D}(f - \nu_k)$ for $k = 0, \ldots, 4$:



But now, since the basis set has fifteen elements, ten shifts remain available to cancel the resulting sidelobes, providing further intuitive support for the already-reasonable notion

[1] An alternative expression for the Dirichlet kernel is perhaps more intuitive: Since $D_M(x) = \frac{1}{2}\sum_{n=-M}^{M} e^{inx}$, the common terms in the two expressions $e^{ix/2}D_M(x)$ and $e^{-ix/2}D_M(x)$ cancel in $e^{ix/2}D_M(x) - e^{-ix/2}D_M(x) = \frac{1}{2}e^{iMx+ix/2} - \frac{1}{2}e^{-iMx-ix/2}$ to yield $D_M(x) = \frac{\sin\left(M+\frac{1}{2}\right)x}{2\sin x/2}$.

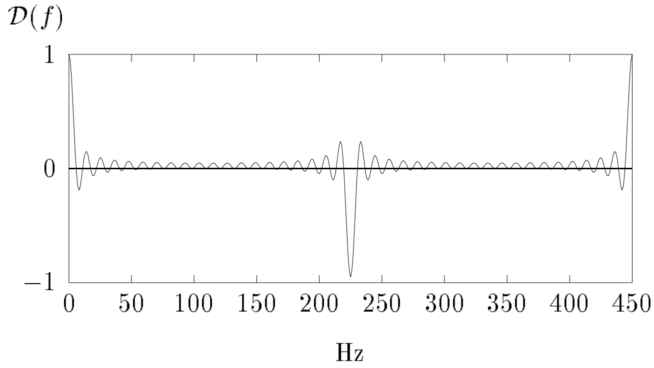
[2] If an equiripple passband is desired, the central passband peaks will actually require slightly higher weights than the outside peaks. This is because of the contributions of the characterizing-function sidelobes.

that a larger passband ripple allows more stopband suppression in an optimal design.

Bear in mind that this is not an actual filter-design procedure. Actual designs should be carried out using optimization software. The arbitrarily shifted characterizing-function view of the design process is simply an intuitive aid to setting up an optimization intelligently, and in the case of nonuniformly spaced taps, to choosing the delays in the first place.

B. Nonuniform Tap Spacing

When tap spacing is nonuniform, characterizing functions can take many forms. If the tap delays are a subset of a uniform grid, the characterizing function will remain periodic. For example, suppose $H(f)$ is a 41-tap halfband filter with taps at $0, \pm T, \pm 3T, \pm 5T, \ldots, \pm 39T$, where $T = \frac{1}{450\,\mathrm{Hz}}$. For this tap placement, the characterizing function $\mathcal{D}(f)$ has a 450 Hz period:

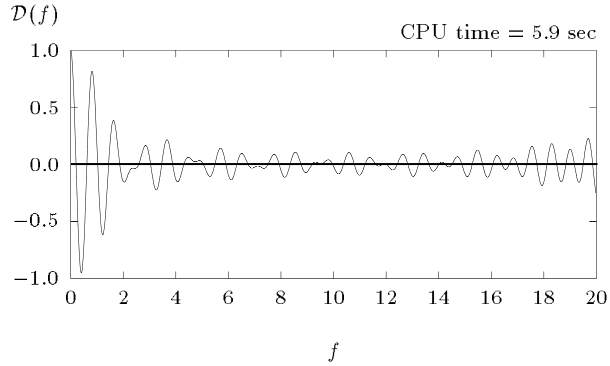


Because this characterizing function $\mathcal{D}(f)$ has the halfband symmetry property that $\mathcal{D}(f) + \mathcal{D}(f - \frac{1}{2T})$ is constant, every frequency response comprising a linear combination of frequency-shifted versions of $\mathcal{D}(f)$ will also be halfband-symmetric. Further, since the shape of $\mathcal{D}(f)$ below the halfband-symmetry point (here 112.5 Hz) is dominated by a single narrow peak, it is easy to imagine that shifts of this function could be made to sum to a response with a passband/stopband structure.

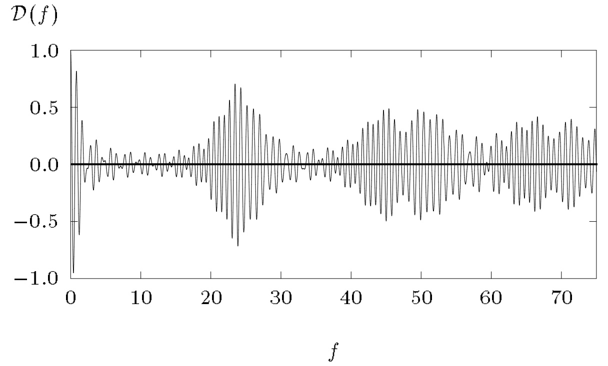
The dominant-peak characteristic is crucial. Without it, optimization to produce a reasonably controlled frequency response can be difficult. Consider this pair of design examples, in which the taps are placed uniformly on a logarithmic time scale and then optimized using a linear program in the FF design language [2; 3]:

Example $\boxed{1}$ Here 10 taps are placed at positive delays uniformly in $\log t$, with 20 taps/octave, and paired taps are placed at corresponding negative delays to ensure linear phase. This FF calculates the characterizing function $\mathcal{D}(f)$:

> Let $N = 10$,
> index h from 1 to N, and
> let $(h, \mathcal{D})$ be a Fourier transform pair.
>
> For each $k = 1, \ldots, N$,
> let h_k be a fixed, real tap pair
> at $\pm 2^{k/20}$ delay, and
> let $h_k = \frac{1}{2N}$.
>
> $\forall f \in [0, 75]$ with $\Delta \le 0.02$, write $\mathcal{D}(f)$ to "response".



A view of $\mathcal{D}(f)$ on a wider frequency scale reveals that the "good" behavior of the function is limited to roughly $|f| \le 20$:



$\boxed{1}$

The overall scale of the DC peak and surrounding sidelobes of $\mathcal{D}(f)$ in Example 1 tells us what frequency ranges we might consider for passband and stopband specifications for a filter based on those tap delays. However, the wild oscillations of the characterizing function near its DC peak can be expected to make optimization of frequency responses difficult, because when placing a shift of $\mathcal{D}(f)$ in a desired place to obtain a peak, say, the optimization must "use up" many of its other shifts of $\mathcal{D}(f)$ to cancel the oscillations of the original shift. Those new shifts introduce more, though smaller, oscillations, which must be canceled, etc.

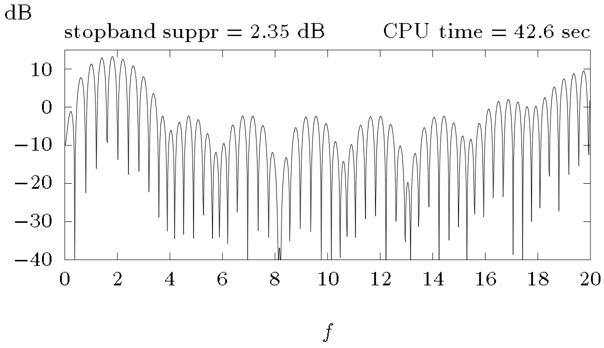
A design example illustrates the hopelessness of this situation.

Example $\boxed{2}$ Here is an attempt, using the tap spacings of Example 1, to optimize a lowpass filter with a passband to $f = 0.35$ and a stopband for $3.5 \le f \le 15$:

> Let $N = 10$,
> index h from 1 to N, and
> let (h, H) be a Fourier transform pair.
>
> For each $k = 1, \ldots, N$,
> let h_k be an optimized, real tap pair at $\pm 2^{k/20}$ delay;
>
> Let δ be optimized and bounded below by 0;
> $\forall f \in [0, 0.35]$ with $\Delta \le 0.02$,
> require $|1 - H(f)| \le 1 - 10^{-10/20}$;
> $\forall f \in [3.5, 15]$ with $\Delta \le 0.02$,
> require $|H(f)| \le \delta$.
>
> Minimize δ, and
> write $-20 \log_{10}(\delta)$ to "StopbandSuppression".

For each $k = 1, \ldots, N$, write h_k to "Weights";
$\forall f \in [0, 100]$ with $\Delta \leq 0.02$, write $H(f)$ to "response".

Even with the extremely wide transition region and the horrendous ± 10 dB passband ripple allowed, the optimization can manage only a pathetic 2.3 dB of stopband suppression:



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III. Design of Characterizing Functions

Arbitrary choices for tap delays fail miserably. How then do we choose a set of tap delays that will produce a well-behaved characterizing function? Probability theory suggests one approach (an approach used routinely in the design of antenna arrays).

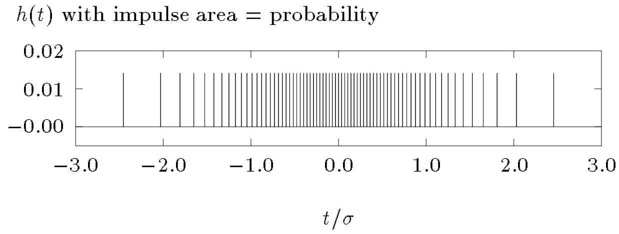
Recall that the characterizing function $\mathcal{D}(f)$ of a set of N delays is just the frequency response associated with uniform tap weights of $1/N$ at those delays. The associated impulse response, say $f(t)$, of such a filter can be interpreted as a probability density function (PDF) of a random variable t. The characteristic function $\phi_t(\omega)$ of this random variable is given by $\phi_t(\omega) = E[e^{i\omega t}] = \int e^{i\omega t} f(t)\, dt = \mathcal{D}\left(\frac{-\omega}{2\pi}\right)$. So, the problem of choosing delays to obtain a good characterizing function amounts to choosing a set of N equiprobable values for random variable t such that t has a well-behaved characteristic function $\phi_t(\omega)$, where well-behaved here means, at the very least, a dominant central lobe and sidelobes that are modest in amplitude, at least over some reasonable interval in ω.

This is an easy task. Every random variable X has a cumulative distribution function (CDF) $F_X(x)$ and an associated characteristic function $\phi_X(\omega) = E[e^{i\omega X}]$ unique to that CDF. The Cramér-Lévy continuity theorem [4, Theorem 6.3.1] states, in part:

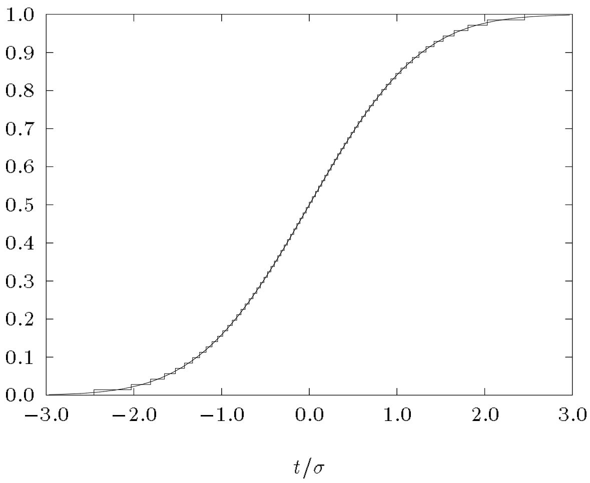
> *If a sequence of CDFs $F_1(x), F_2(x), \ldots$ converges to some CDF $F(x)$ at every x at which $F(x)$ is continuous, then the characteristic functions $\phi_1(\omega), \phi_2(\omega), \ldots$ of the $F_1(x), F_2(x), \ldots$ converge to $\phi(\omega)$, the characteristic function of $F(x)$, uniformly on any finite interval.*

Thus we can simply choose a desirable characteristic function $\phi(\omega)$ and choose delays so that our PDF $f(t)$ integrates to a CDF $F_t(t)$ that closely approximates the CDF $F(t)$ associated with $\phi(\omega)$. If we approximate $F(t)$ well enough, we can make $\phi_t(\omega)$ approximate $\phi(\omega)$ as closely as desired over any finite interval we like. Equivalently, we can make $\mathcal{D}(f)$ approximate $\phi(-2\pi f)$ arbitrarily closely. Here is an example:

Example $\boxed{3}$ Suppose we wish $\mathcal{D}(f)$ to approximate $\phi(-2\pi f) = e^{-f^2/2\sigma_f^2}$, a Gaussian frequency response with characteristic width σ_f in frequency. Then $\phi(f) = e^{-\frac{1}{2}\left(\frac{f}{2\pi\sigma_f}\right)^2}$, the characteristic function of a $N(0, \sigma^2)$ Gaussian random variable with $\sigma = \frac{1}{2\pi\sigma_f}$. To choose N delays so that the filter's stairstep unit-step response approximates the CDF of a $N(0, \sigma^2)$ random variable, place a delay at time $\sigma\Phi^{-1}\left(\frac{m+1/2}{N}\right)$ for each $m = 0, \ldots, N-1$, where $\Phi(x)$ is the CDF of a $N(0,1)$ random variable. For $N = 71$, this results in this impulse-response/PDF $h(t)$:



and this unit-step-response/CDF:



The CDF associated with $h(t)$ is, of course, the stairstep function. The smooth curve plotted underneath it is the CDF $\Phi(t/\sigma)$ being approximated.

This FF reads the normalized (to $\sigma = 1$) delays from file "levels" and calculates the characterizing function $\mathcal{D}(f)$ for $\sigma_f = 100$ Hz:

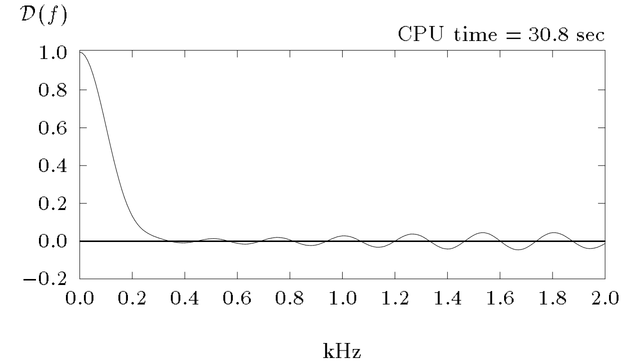
Let Hz = 1, and let kHz = 10^3 Hz.

Let $M = 35$, and let $N = 2M + 1$.
Index h from 0 to M, and
let $(h, \mathcal{D})$ be a Fourier transform pair.
Interpolate QuantLevel from real file "levels"
with spacing 1 from $-M$.

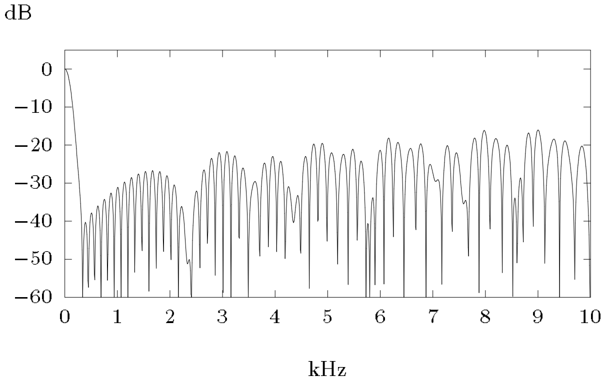
Let $\sigma = \frac{1}{2\pi \times 100\text{ Hz}}$;
let h_0 be a fixed, real tap at $\sigma \times$ QuantLevel(0) delay;
for each $k = 1, \ldots, M$,
let h_k be a fixed, real tap pair
at $\pm\sigma \times$ QuantLevel(k) delay;
for each $k = 0, \ldots, M$, let $h_k = \frac{1}{N}$.

$\forall f \in [0, 120\,\text{kHz}]$ with $\Delta \leq 10$ Hz,
write $\mathcal{D}(f)$ to "response".

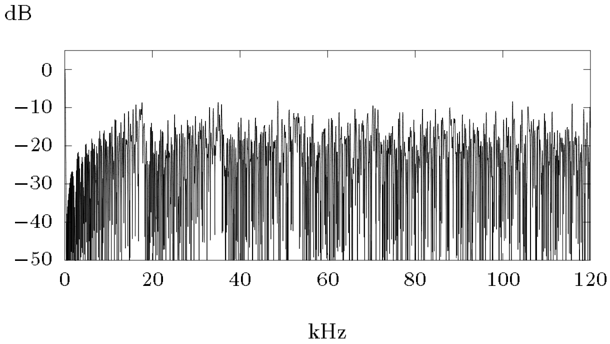
Over a frequency range of several σ_f, the resulting $\mathcal{D}(f)$ is well-behaved, with an obvious Gaussian resemblance in the main lobe, as this linear-scale plot shows:



Over a wider frequency range however, this $\mathcal{D}(f)$ has random-looking sidelobes down about 20 dB.



On an even wider scale, the sidelobes peak around −10 dB and $\mathcal{D}(f)$ loses all appearance of structure:



The first of these three views of $\mathcal{D}(f)$ suggests that optimizing a frequency response based on these delays should be straightforward with passband and stopband widths on the order of a few hundred Hertz.

This FF tests this notion by optimizing a 71-tap lowpass with a ±0.1 dB passband to 500 Hz and a stopband for 1 kHz $\leq f \leq$ 10 kHz:

Let Hz = 1, and let kHz = 10^3 Hz.

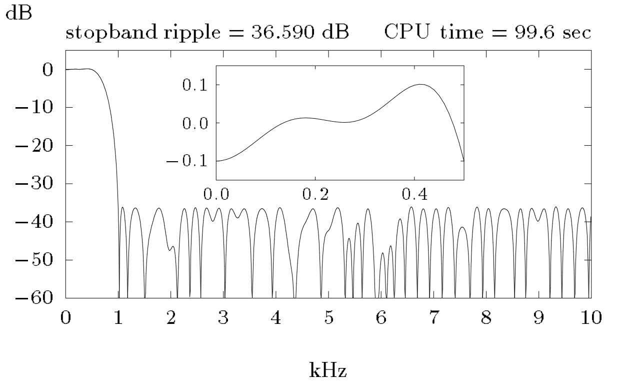
Let $M = 35$, index h from 0 to M, and
let (h, H) be a Fourier transform pair.
Interpolate QuantLevel from real file "levels"
with spacing 1 from $-M$.

Let $\sigma = \frac{1}{2\pi \times 100 \text{ Hz}}$,
let h_0 be an optimized, real tap
at $\sigma \times$ QuantLevel(0) delay, and
for each $k = 1, \ldots, M$,
let h_k be an optimized, real tap pair
at $\pm\sigma \times$ QuantLevel(k) delay.

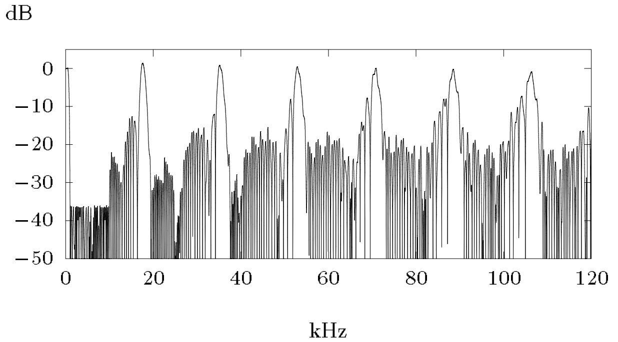
Let δ be optimized and bounded below by 0;
$\forall f \in [0, 0.5\,\text{kHz}]$ with $\Delta \leq 50$ Hz,
require $|1 - H(f)| \leq 1 - 10^{-0.1/20}$;
$\forall f \in [1\,\text{kHz}, 10\,\text{kHz}]$ with $\Delta \leq 50$ Hz, require $|H(f)| \leq \delta$.

Minimize δ, and
write $-20\log_{10}(\delta)$ to "StopbandSuppression".

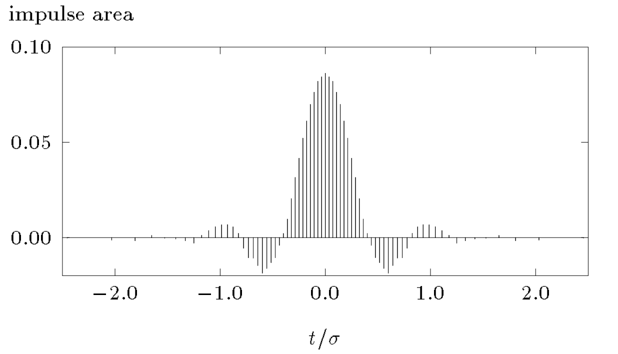
For each $k = 0, \ldots, M$, write h_k to "Weights";
$\forall f \in [0, 120\,\text{kHz}]$ with $\Delta \leq 10$ Hz,
write $H(f)$ to "response".



The tap weights appear to give some structure to the frequency response on a wide scale, but it is clearly not periodic as with uniformly spaced taps:



Finally, here is the impulse response of the optimized filter:



$\boxed{3}$

IV. Summary

The family of frequency responses that can be generated by adjusting the weights on a tapped-delay-line filter with nonuniformly spaced taps is just the span of a basis set whose elements are *arbitrary* (as long as linear independence is preserved) frequency shifts of a *characterizing function* of the set of delays. This characterizing function is just the unity-DC-gain frequency response that results when uniform weights are used at those delays. If the delays have been chosen to provide a unimodal characterizing function with low sidelobes (probability theory provides one approach), at least over the frequency range of interest, then it will be relatively easy to approximate a desired frequency response from linear combinations of various frequency shifts of the characterizing function, or equivalently, an optimal design of the tap weights will yield good results. If the characterizing function contains wild, high-level oscillations and/or many large sidelobes, the approximation problem is less well-conditioned, and it may not be possible to get the ill-behaved, weighted characterizing-function shifts to sum to a desired response, even with optimization techniques.

The importance of the characterizing-function idea lies fundamentally

1. in the intuition it gives the designer as to whether the available number of degrees of freedom (dimensionality or number of taps) will be adequate to the design task at hand and
2. in the insight it provides to guide the choice of the delays.

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Dr. Coleman is a member of the IEEE, Eta Kappa Nu, Tau Beta Pi, and Sigma Xi.