

The Spectral Products Created by Nonlinear Intersymbol Interference in NRZ Data

Jeffrey O. Coleman

Naval Research Laboratory (<http://www.nrl.navy.mil/>)

Radar Division, Signal Processing Theory & Methods Section*

Abstract — In two-level noise-shaped D/A conversion, a delta-sigma modulator quantizes a bandlimited discrete-time signal into a binary drive for a two-level D/A converter that then outputs a nominally clean NRZ data waveform with the quantization-noise spectrum shaped to be very low in the signal band. Nonlinear intersymbol interference in the final two-level conversion can, however, corrupt the signal band with intermodulation noise. Here a nonlinear model of that two-level conversion is used to derive its output spectrum for test data that are either periodic or random with i.i.d. bits, results that can enable laboratory measurement of model parameters and guide needed circuit improvements.

I. INTRODUCTION

When BPSK pulse modulation or filtered two-level D/A conversion applies pulse shape $p(t)$ in a linear, shift-invariant (LSI) way to random data sequence $d(n)$ to create $y(t) = \sum_n d(n)p(t - nT)$, ordinary asynchronous spectral measurement estimates the continuous-time power spectral density of $y(t - t_0)$ given by

$$S_y(f) = S_d(fT) \frac{|P(f)|^2}{T} \quad (1)$$

(see Section A), where random delay t_0 is uniform on $[0, T)$ and independent of other variables. Here $S_d(\nu)$ is the power spectral density of the data, and $p(t) \leftrightarrow P(f)$ is a continuous-time Fourier pair. In noise-shaped two-level D/A conversion, a $\Delta\Sigma$ modulator creates a ± 1 valued data sequence $d(n) = s(t) + n(t)$ with signal $s(t)$ and quantization noise $n(t)$ uncorrelated and corresponding to spectrally nonoverlapping $S_d(\nu)$ components so that $P(f)$ can eliminate the noise while preserving the signal. At high clock rates, however, circuit nonlinearities often invalidate this model and create signal-band intermodulation noise.

Intersymbol interference (ISI) without qualification usually means linear ISI that can be modeled in $p(t) \leftrightarrow P(f)$. *Nonlinear ISI* is data-related distortion that cannot be so modeled. (Linear and nonlinear ISI are collectively termed *deterministic jitter* by some component makers.) Here a variation of the nonlinear-ISI model in [1] is used to obtain three spectral results.

1. The model's output spectrum in terms of (a) the waveforms that parameterize the model and (b) easily estimated (in simulation) spectra and cross spectra of the data and various parity bits taken over the recent data history.
2. Spectra and cross spectra (b) when the $\Delta\Sigma$ modulator output is replaced with periodic test data. This enables lab measurement of spectral properties of waveforms (a).

3. Those key discrete-time spectra when the $\Delta\Sigma$ modulator output is replaced with a source of data that are random, independent, and equiprobably ± 1 . In at least one key case, that of mismatched rise and fall time constants for a nominally NRZ signal, this allows parameterizing waveforms (a) to be derived exactly using a crosscorrelation. The insight provided by such an analytic solution can guide implementation improvement in the laboratory. (Due to space constraints the mismatched rise/fall derivation itself is omitted here. It will be presented elsewhere.)

II. THE NONLINEAR CONVERSION MODEL

This section recaps the nonlinear model of single-bit D/A conversion presented in [1]. Here it is both a little simpler and a little more general, and motivational material is largely omitted.

A. D/A Convolution

Define rate $1/T$ D/A convolution of a discrete-time signal $\mathbf{d}(n)$ and a continuous-time signal $\mathbf{c}(t)$ by

$$\begin{aligned} (\mathbf{d} \star \mathbf{c})(t) &\triangleq T \sum_n \mathbf{d}(n) \mathbf{c}(t - nT) \\ (\mathbf{c} \star \mathbf{d})(t) &\triangleq T \sum_n \mathbf{c}(t - nT) \mathbf{d}(n). \end{aligned} \quad (2)$$

D/A convolution is not assumed commutative, as $\mathbf{d}(n)$ and $\mathbf{c}(t)$ may be dimensionally compatible matrices or vectors rather than scalars. The respective Fourier transforms are

$$\begin{aligned} T \sum_n \mathbf{d}(n) \mathbf{C}(f) e^{-j2\pi n f T} &= T \mathbf{D}(fT) \mathbf{C}(f) \\ T \sum_n \mathbf{C}(f) e^{-j2\pi n f T} \mathbf{d}(n) &= \mathbf{C}(f) T \mathbf{D}(fT). \end{aligned}$$

B. A Nonlinear Model of Two-Level D/A Conversion

The system in Fig. 1 realizes a general time-invariant, causal map with finite nonlinear memory from a two-level discrete-time signal input $d(n)$ to a continuous-time analog output $y(t)$. A signal input $d(n)$, not shown, and the $\{0, 1\}$ -valued control input $x(n)$ on the lower right are affine (linear plus a constant) in each other. Signal input $d(n)$ is the value to be converted, and control input $x(n)$ is a $\{0, 1\}$ digital encoding of that value. The encoding determines conversion type. A few possibilities:

conversion type	$d(n) \in$	mapping
bipolar	$\{\pm 1\}$	$d(n) = 2x(n) - 1$
unipolar	$\{0, 1\}$	$d(n) = x(n)$
bipolar, inverting	$\{\pm 1\}$	$d(n) = 1 - 2x(n)$
unipolar (noninv), scaled by two	$\{0, 2\}$	$d(n) = 2x(n)$

* This work was supported by the NRL funding base and the Office of Naval Research (<http://www.onr.navy.mil/>). Author contact: <http://alum.mit.edu/www/jeffc>

Matrix $R_{\mathbf{u}}(k)$ in (10) is cyclic for each k and therefore diagonalized by an M -point DFT matrix \mathbf{D} with $\mathbf{D}_{m,n} = e^{-j2\pi nm/M}$:

$$\begin{aligned} [\mathbf{D}R_{\mathbf{u}}(k)\mathbf{D}^H]_{m,n} &= \frac{1}{M} \sum_i \mathbf{D}_{m,i} \mathbf{D}_{n,i+k}^* \\ &= \frac{1}{M} \sum_i e^{-j2\pi(mi-n(i+k))/M} \\ &= e^{j2\pi \frac{nk}{M}} \frac{1}{M} \sum_i e^{-j2\pi \frac{mi}{M}} \left(e^{-j2\pi \frac{ni}{M}} \right)^* \\ &= e^{j2\pi \frac{nk}{M}} \frac{1}{M} [\mathbf{D}\mathbf{D}^H]_{m,n}. \end{aligned}$$

But $\frac{1}{M}\mathbf{D}^H$ represents an inverse DFT, so $\frac{1}{M}\mathbf{D}\mathbf{D}^H = \mathbf{I}$ and $\mathbf{D}R_{\mathbf{u}}(k)\mathbf{D}^H$ is diagonal with $[\mathbf{D}R_{\mathbf{u}}(k)\mathbf{D}^H]_{i,i} = e^{j2\pi ik/M}$.

Getting the scale factors right in the Fourier transform $S_{\mathbf{u}}(fT)$ is worth some risk of belaboring the obvious. By definition,

$$S_{\mathbf{u}}(fT) = \sum_k R_{\mathbf{u}}(k) e^{-j2\pi kfT} \quad (11)$$

$$= \int \sum_k R_{\mathbf{u}}(k) \delta(t-kT) e^{-j2\pi ft} dt, \quad (12)$$

so we can certainly write

$$\begin{aligned} &[\mathbf{D}S_{\mathbf{u}}((f + \frac{i}{MT})T)\mathbf{D}^H]_{i,i} \\ &= \int \sum_k [\mathbf{D}R_{\mathbf{u}}(k)\mathbf{D}^H]_{i,i} e^{-j2\pi \frac{i}{MT}t} \delta(t-kT) e^{-j2\pi ft} dt \\ &= \int \sum_k \delta(t-kT) e^{-j2\pi ft} dt, \end{aligned}$$

and the area of the DC line of $[\mathbf{D}S_{\mathbf{u}}((f + \frac{i}{MT})T)\mathbf{D}^H]_{i,i}$ or the $f = \frac{i}{MT}$ line of $[\mathbf{D}S_{\mathbf{u}}(fT)\mathbf{D}^H]_{i,i}$ is the average value of $\sum_k \delta(t-kT)$, or just $\frac{1}{T}$. Using $\frac{1}{M}\mathbf{D}^H\mathbf{D} = \mathbf{I}$ then,

$$TS_{\mathbf{u}}(fT) = \frac{1}{M^2} \mathbf{D}^H \mathbf{\Lambda}(f) \mathbf{D},$$

where diagonal matrix function $\mathbf{\Lambda}(f)$ has period $1/T$ and

$$\mathbf{\Lambda}(f) = \begin{bmatrix} \delta(f) & & & \\ & \delta(f - \frac{1}{MT}) & & \\ & & \ddots & \\ & & & \delta(f - \frac{M-1}{MT}) \end{bmatrix}$$

for $0 \leq f < \frac{1}{T}$. The autocorrelation $R_{\mathbf{u}}(k)$ of vector $\mathbf{u}(n)$ is stationary, so the output power spectrum is, from (4) and (8),

$$\begin{aligned} S_y(f) &= \mathbf{G}(f) \mathbf{A}_p T S_{\mathbf{u}}(fT) \mathbf{A}_p^H \mathbf{G}^H(f) \\ &= \frac{1}{M^2} (\mathbf{G}(f) \mathbf{A}_p \mathbf{D}^H) \mathbf{\Lambda}(f) (\mathbf{G}(f) \mathbf{A}_p \mathbf{D}^H)^H \\ &= \sum_i \delta(f - \frac{i}{MT}) \left| \mathbf{G}(f) \frac{1}{M} [\mathbf{A}_p \mathbf{D}^H]_{\text{column } i} \right|^2. \end{aligned}$$

The discrete-time drive has period M and so yields an output spectrum of period $1/T$ with each period containing M equally spaced lines. In the above sum, there are M period $1/T$ subfamilies of lines, each containing one line per period. Each subfamily has associated with it a transfer function that determines its lines' areas. The M transfer functions corresponding to the M subfamilies are the M elements of row vector $\frac{1}{M} \mathbf{G}(f) \mathbf{A}_p \mathbf{D}^H$.

C. The Spectrum Produced by a Random Input

Let \mathcal{E}_n^i denote the event $[\mathbf{N}(\mathbf{x}(n)) = \hat{e}_i]$ so that

$$R_{\mathbf{N}}(k) = \sum_{i,j} \hat{e}_i \hat{e}_j^T P(\mathcal{E}_{n-k}^j \cap \mathcal{E}_n^i).$$

When $N=4$, $n=9$, and $k=1$ for example, the Fig. 1 delay line maps $x(n)$ to elements of vectors $\mathbf{x}(n)$ and $\mathbf{x}(n-k)$ like this:

element of $\mathbf{x}(9)$:	0	1	2	3			
... $x(3)$	$x(4)$	$x(5)$	$x(6)$	$x(7)$	$x(8)$	$x(9)$...
element of $\mathbf{x}(8)$:	0	1	2	3			

The first three elements of $\mathbf{x}(9)$ are necessarily also the last three elements of $\mathbf{x}(8)$. Generalizing, event $\mathcal{E}_{n-k}^j \cap \mathcal{E}_n^i$ can be nonnull only when the two subvectors

elements $\max(0, -k)$ through $\min(N-1, N-1-k)$ of \mathbf{x}_i
elements $\max(0, k)$ through $\min(N-1, N-1+k)$ of \mathbf{x}_j ,

are equal, where binary vector \mathbf{x}_i is one of the possible values of address vector $\mathbf{x}(n)$ and is defined by $2^T \mathbf{x}_i = i$. Event $\mathcal{E}_{n-k}^j \cap \mathcal{E}_n^i$ then corresponds to a specific $\min(2N, N+|k|)$ bits of control input $x(n)$ matching particular values found in various elements of \mathbf{x}_i and \mathbf{x}_j . If random control input $x(n)$ has independent and equiprobably binary-valued samples, then

$$[R_{\mathbf{N}}(k)]_{ij} = \begin{cases} \frac{1}{2^{\min(2N, N+|k|)}} & \text{for } (i, j, k) \text{ yielding} \\ & \text{equal subvectors,} \\ 0 & \text{otherwise.} \end{cases}$$

When $|k| \geq N$ the subvectors have length zero and so are automatically equal, and the 2^{-2N} value given here is valid.

When else are the subvectors equal? We have four ways to express each of the two events of interest:

$$\begin{array}{c|c} \mathcal{E}_n^i & \mathcal{E}_{n-k}^j \\ \hline [\mathbf{N}(\mathbf{x}(n)) = \hat{e}_i] & [\mathbf{N}(\mathbf{x}(n-k)) = \hat{e}_j] \\ [\mathbf{x}(n) = \mathbf{x}_i] & [\mathbf{x}(n-k) = \mathbf{x}_j] \\ [2^T \mathbf{x}(n) = i] & [2^T \mathbf{x}(n-k) = j] \end{array}$$

The subvector condition above used the third of these, but there is advantage now in using the fourth instead to map each possible binary address vector to an integer. The matching-subvector condition above becomes the requirement that, if $M(\ell, m)$ and $L(\ell, m)$ respectively denote the m most- and least-significant bits of the address vector that maps to integer ℓ ,

$$\begin{array}{ll} \text{for } -N < k < 0, & M(j, N+k) = L(i, N+k); \\ \text{for } 0 < k < N, & L(j, N-k) = M(i, N-k); \\ \text{for } k = 0, & \text{either condition above or just } i = j. \end{array}$$

For $0 \leq k \leq N$ we can write i and j in terms of their bits in common and their other bits as

$$\begin{aligned} j &= 2^{N-k} j_o + b_c \text{ with } 0 \leq b_c < 2^{N-k} \\ i &= 2^k b_c + i_o \text{ with } 0 \leq i_o < 2^k \end{aligned}$$

so that

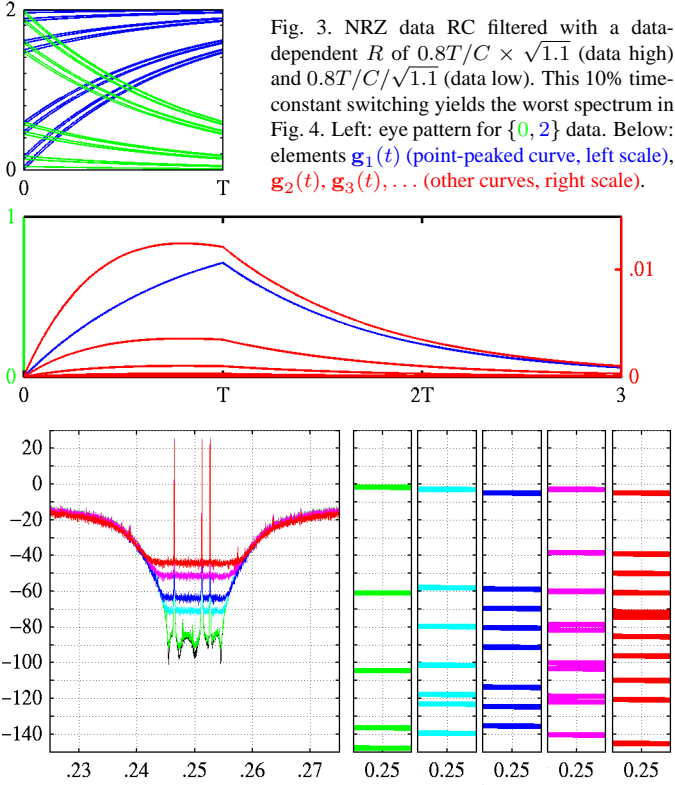


Fig. 3. NRZ data RC filtered with a data-dependent R of $0.8T/C \times \sqrt{1.1}$ (data high) and $0.8T/C/\sqrt{1.1}$ (data low). This 10% time-constant switching yields the worst spectrum in Fig. 4. Left: eye pattern for $\{0, 2\}$ data. Below: elements $\mathbf{g}_1(t)$ (point-peaked curve, left scale), $\mathbf{g}_2(t)$, $\mathbf{g}_3(t)$, \dots (other curves, right scale).

Fig. 4. Large plot on left: bandpass $\Delta\Sigma$ spectrum $T^{-1}S_y(f)$ for three signal tones in a signal band with width 1% of clock rate $1/T$. Each of five plots on right: signal-band levels of $T^{-1}\mathbf{G}_i(f)$ for $i = 1, 2, \dots$. The bottom of the six curves on the left (almost buried under the next curve) has identical data-high and data-low RC time constants $\tau_{hi} = \tau_{low} = 0.2T$. For the other five curves on the left, bottom to top, and the five plots on the right, left to right:

$\frac{\sqrt{\tau_{hi}\tau_{low}}}{(\tau_{hi} - \tau_{low})/\tau_{low}}$	0.2T	0.4T	0.8T	0.4T	0.8T
	1%	1%	1%	10%	10%

$$R_{\mathbf{N}}(k) = \frac{1}{2^{\min(2N, N+k)}} \sum_{b_c=0}^{2^{N-k}-1} \left(\sum_{i_o=0}^{2^{k-1}} \hat{e}_i \right) \left(\sum_{j_o=0}^{2^{k-1}} \hat{e}_j \right)^T$$

For $k > N$ the value of $R_{\mathbf{N}}(k)$ is constant, fixed at $R_{\mathbf{N}}(N)$. For negative k , either replace k on the right above with $-k$ and swap i and j or use the conjugate symmetry $R(k) = R^H(-k)$ required of all autocorrelations to obtain $R_{\mathbf{N}}(k)$ from its value for positive k . These yield identical results.

In the $N=2$ case, for example, $R_{\mathbf{N}}(k)$ is

$$\begin{array}{ccccc} k \leq 2 & k = -1 & k = 0 & k = 1 & k \geq 2 \\ \frac{1}{16} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix} & \frac{1}{8} \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix} & \frac{1}{4} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} & \frac{1}{8} \begin{bmatrix} 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 \end{bmatrix} & \frac{1}{16} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix} \end{array}$$

For each k the matrix elements sum to unity, because they are probabilities of events that partition the sample space.

Codeword sequence $\mathbf{a}(n)$ of (3) has autocorrelation

$$R_{\mathbf{a}}(k) = \mathbf{A} R_{\mathbf{N}}(k) \mathbf{A}^H.$$

Using (6) with $\mathbf{C} = \mathbf{C}_{\{0,2\}}$, this effects a remarkable simplification of $R_{\mathbf{N}}(k)$. Not proven here but easily verified computationally for all N likely to be useful in the foreseeable future,

$$R_{\mathbf{a}}(k) = \mathbf{1} \mathbf{1}^T + (\mathbf{I} - \hat{e}_0 \hat{e}_0^T) \delta_k, \quad (13)$$

where vector $\mathbf{1}$ is $2^{N-1} + 1$ ones, \mathbf{I} is a $2^{N-1} + 1$ square identity matrix, and here \hat{e}_0 is the length $2^{N-1} + 1$ unit vector $[1 \ 0 \ \dots \ 0]^T$.

Spectral density $S_{\mathbf{a}}(fT)$ now follows from (12). The first term, $\sum_k \mathbf{1} \mathbf{1}^T \delta(t - kT)$, is supported on $T\mathbb{Z}$ and has period T , so its Fourier transform has period $1/T$ and is supported on $T^{-1}\mathbb{Z}$ and must be a periodic impulse train. Each area is just that of the DC impulse and so given by the average over t of $\sum_k \mathbf{1} \mathbf{1}^T \delta(t - kT)$ or just $\frac{1}{T} \mathbf{1} \mathbf{1}^T$. Direct evaluation of (12) yields the Fourier transform of $\sum_k (\mathbf{I} - \hat{e}_0 \hat{e}_0^T) \delta_k \delta(t - kT)$, so periodically extending $\delta(f)$ to $\delta_{1/T}(f) \triangleq \sum_{\ell} \delta(f - \ell/T)$,

$$TS_{\mathbf{a}}(fT) = \delta_{1/T}(f) \mathbf{1} \mathbf{1}^T + T(\mathbf{I} - \hat{e}_0 \hat{e}_0^T).$$

Finally, from (8),

$$\begin{aligned} S_y(f) &= \mathbf{G}(f) TS_{\mathbf{a}}(fT) \mathbf{G}^H(f) \\ &= \delta_{1/T}(f) \left| \sum_i \mathbf{G}_i(f) \right|^2 + T \sum_{i \neq 0} |\mathbf{G}_i(f)|^2. \end{aligned}$$

The first term contributes clock-harmonic lines, and the second term is white and contributed to by all but the $\mathbf{G}_{\text{clock}}(f)$ component of transformed waveform row vector $\mathbf{G}(f)$.

The fact that the various $|\mathbf{G}_i(f)|^2$ for $i = 2, 3, \dots$ affect the output with the same scaling suggests that they can be compared to get a rough sense of how the various nonlinear terms will contribute to a system output. This is next explored numerically.

IV. COMPUTATION AND CONCLUSIONS

Suppose a system subjects NRZ data (noninverting unipolar conversion scaled by two) to an RC time constant that is a function of the current data value, giving its output mismatched rise and fall times. Using crosscorrelation (9), the waveform row vector for this system was derived exactly using $\mathbf{A} = \mathbf{C}_{\{0,2\}} \mathbf{X} \mathbf{H}$ in the model. (The details are omitted here for space reasons but will be presented elsewhere.) It is plotted in Fig. 3 (without $\mathbf{g}_0(t)$, as clock harmonics are of little interest in $\Delta\Sigma$ systems) for particular but unremarkable parameters. The huge linear ISI contribution visible in the eye pattern is nearly irrelevant to $\Delta\Sigma$ performance, but the tiny nonlinear-ISI component, visible as slight asymmetry top to bottom and resulting from $\mathbf{g}_2(t)$, raises the noise floor some 40 dB in the associated $\Delta\Sigma$ spectrum in Fig. 4 (left, top curve), completely destroying signal-band SNR. The other Fig. 4 curves suggest—clearly more work is needed here—that $\mathbf{G}_3(f)$ (rather than the larger $\mathbf{G}_2(f)$) dominates the nonlinear spectral contributions. This term contributes much more when time constants are long, so much more mismatch is tolerated when time constants are short.

REFERENCES

- [1] J. O. Coleman, "Modeling Finite-Memory Nonlinearity in Unit DAC Elements, Binary Storage Channels, and BPSK Data Channels," in *17th (IEEE) European Conf. on Circuit Theory and Design* (Cork, Ireland), Aug. 29–Sep. 1 2005.
- [2] V. K. Prabhu and H. E. Rowe, "Spectra of digital phase modulation by matrix methods," *Bell System Technical J.*, vol. 53, pp. 899–935, May 1974.