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The Fundamental Input/Output Structure of a Linear, Time-Varying Array Receiver

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Abstract—*A single-output array receiver based on static antenna elements of arbitrary geometry that drive linear, time-invariant signal processing is well known to be fully characterized by a frequency response dependent on direction-of-arrival (DOA). This paper develops the generalized characterization that results when the time-invariance requirement is removed. The special case based on periodically varying signal processing is further derived and shows such a system to be fundamentally, structurally incapable of realizing a long-sought goal in navigation engineering: a frequency shift smoothly dependent on DOA. Self-contained developments of time-varying continuous-time linear systems and of a 4D Fourier-transform view of propagating electromagnetic waves are given in the appendices.*

1. Introduction

The unpredictable variety of incident signals generally forces an array-antenna system to either behave **L**inearly or risk being overwhelmed by spurious intermodulation products. Conventional arrays are **T**ime **I**nvariant as well. If we include the combining of element outputs into one signal in the term "array," an **LTI** array behaves mathematically as a multidimensional-input filter that samples incident plane waves at the spatial origin, subjects the resulting sinusoids to gains and phase shifts determined by frequencies and directions of arrival (DOAs), and sums these modified sinusoids to form an output. The array pattern is the gain/phase-shift as determined by DOA and is generally a function of frequency. How does this mathematical structure generalize if we remove the time-invariance requirement?

For historical reasons time-varying arrays (excluding conventional, slow beamsteering) have been considered primarily for navigation. The **VOR** (**V**HF **O**mni**r**ange) system widely used for aircraft navigation prior to the GPS era, for example, used a time-varying transmit array. But there are time-varying receive arrays as well. Baghdady [1], for example, presents systems creating "synthetic antenna motion" by linearly combining element outputs with rapidly varying "blending functions." The rapid movement of the array's phase center imparts a DOA-dependent phase (or frequency) modulation to incoming signals. The navigation community has long sought element geometry and blending functions to create a frequency shift that is a smooth function of DOA. A practical result of this paper is to prove that this Holy Grail will not be found.

The heart of this paper is in the Section 2 development from first principles of the mathematical input/output structure of a linear, time-varying array receiver, subsequently specialized to periodic signal processing in Section 3. These developments use mathematically convenient characterizations of time-varying linear systems and of electromagnetic waves as developed in the concise reviews in Appendices A and B respectively. These appendices should be read first.

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2. The Time-Varying Array Pattern

A time-varying array of stationary elements is fully modeled by a particular cascade of three linear component systems.

The first component system models the LTI mapping from the incident field $U(\ell, f)$ in the absence of the physical array elements to the electric potential on the antenna structure itself. The potential field, normalized to be dimensionless, is given by $P = G \star U$, where $G(\ell, \eta; f, \nu) = H(\ell, \eta; f)\, \delta(\nu - f)$, in the spirit of (A.12), reflects the time invariance of this element-hardware system. So

$$P(\ell, f) = \int_{\mathbb{R}^3} H(\ell, \eta; f)\, U(\eta, f)\, d\eta.$$

The second component system simply samples this potential at a finite collection $\mathcal{X}$ of spatial locations, the feedpoints of the transmission lines. Temporal frequency dependence (here flat) is modeled as in Equation (A.12), and multiplication by a sampling function $s(\mathbf{x})$ of spatial impulses is modeled as in (A.13). Impulse areas in $V = R \star P$ then, where $R(\ell, \eta; f, \nu) = S(\ell - \eta)\, \delta(\nu - f)$, represent element output voltages, and so

$$V(\ell,f)=\int_{\mathbb{R}^3} S(\ell-\eta)\,P(\eta,f)\,d\eta,$$

where $S(\ell)=\int s(\mathbf{x})\,\mathrm{e}^{-\mathrm{j}2\pi\ell\cdot\mathbf{x}}\,d\mathbf{x}$ or

$$\begin{aligned} S(\ell) &= \int \Big[\sum_{x_{\mathrm{elt}}\in\mathcal{X}} \delta(\mathbf{x}-\mathbf{x}_{\mathrm{elt}})\Big]\,\mathrm{e}^{-\mathrm{j}2\pi\ell\cdot\mathbf{x}}\,d\mathbf{x} \\ &= \sum_{x_{\mathrm{elt}}\in\mathcal{X}} \mathrm{e}^{-\mathrm{j}2\pi\ell\cdot\mathbf{x}_{\mathrm{elt}}}\,. \end{aligned} \tag{1}$$

Finally, we suppose that these voltages obtained at the element outputs are captured and processed by a third component system that is both linear *and* time varying. The input to this system is formally a function of three spatial dimensions and one temporal dimension, but the output is simply a time function. The simplest way to model this mathematically is to follow a space invariant system by sampling at the spatial origin, which of course corresponds to integrating the output spectrum over all spatial frequencies. This retains the flexibility needed to independently model the coupling from each input location in space to the sampled output. So, the frequency response of this space-invariant system representing the signal-processing-based element feedsystem can be written as $F(\ell,\eta;f,\nu)=D(\ell;f,\nu)\,\delta(\eta-\ell)$, leading to output

$$\begin{aligned} Y(f) &= \int (F\star V)(\ell,f)\,d\ell \\ &= \iint D(\ell;f,\nu)\,V(\ell,\nu)\,d\nu\,d\ell. \end{aligned}$$

These three components together then yield system behavior

$$\begin{aligned} Y(f)=\iiiint\ & D(\mathbf{b}\ ;f,\nu) \\ &\times S(\mathbf{b}-\mathbf{a}) \\ &\times H(\mathbf{a},\ \ell;\ \nu) \\ &\times U(\ell,\ \nu) \\ &\ d\mathbf{b}\ d\mathbf{a}\ d\ell\ \ d\nu. \end{aligned}$$

(Repeated variables in the integrand are to be integrated out, in the spirit of a tensor product.) Substituting (1) and rearranging,

$$Y(f)=\sum_{x_{\mathrm{elt}}\in\mathcal{X}}\int_{\mathbb{R}} \overbrace{\int_{\mathbb{R}^3} D(\mathbf{b};f,\nu)\ \mathrm{e}^{-\mathrm{j}2\pi\mathbf{b}\cdot\mathbf{x}_{\mathrm{elt}}}\ d\mathbf{b}}^{D_{\mathbf{x}_{\mathrm{elt}}}(f,\nu)\,\triangleq\,\cdot} \int_{\mathbb{R}^3} \underbrace{\int_{\mathbb{R}^3} \mathrm{e}^{\mathrm{j}2\pi\mathbf{a}\cdot\mathbf{x}_{\mathrm{elt}}}\ H(\mathbf{a},\ell;\nu)\,d\mathbf{a}}_{H_{\mathbf{x}_{\mathrm{elt}}}(\ell,\nu)\,\triangleq\,\cdot}\ U(\ell,\nu)\,d\ell\,d\nu, \tag{2}$$

which is now concisely written as

$$Y(f)=\sum_{x_{\mathrm{elt}}\in\mathcal{X}}\int_{\mathbb{R}} D_{\mathbf{x}_{\mathrm{elt}}}(f,\nu)\int_{\mathbb{R}^3} H_{\mathbf{x}_{\mathrm{elt}}}(\ell,\nu)\,U(\ell,\nu)\,d\ell\,d\nu. \tag{3}$$

Here $D_{\mathbf{x}_{\mathrm{elt}}}(f,\nu)$ can be interpreted as the frequency response of the time-varying linear signal-processing system associated with the element feedpoint at $\mathbf{x}_{\mathrm{elt}}$. Likewise $H_{\mathbf{x}_{\mathrm{elt}}}(\ell,f)$ can be interpreted as the frequency response of a linear, time-invariant, space-invariant system that characterizes the pattern of the element fed at $\mathbf{x}_{\mathrm{elt}}$.

2.1 Element Characterization

An alternative to the definition of $H_{\mathbf{x}_{\mathrm{elt}}}$ in (2) can be developed from the nature of the elements represented. For example, the simplest element pattern $H_{\mathbf{x}_{\mathrm{elt}}}(\ell,f)$ is isotropic. In this case, it is helpful to temporarily re-express the inner integral in (3), the element output spectrum, in terms of functions inverse Fourier transformed on the spatial variable only:

$$\int_{\mathbb{R}^3} H_{\mathbf{x}_{\mathrm{elt}}}(\ell,\nu)\,U(\ell,\nu)\,d\ell=\int_{\mathbb{R}^3} h_{\mathbf{x}_{\mathrm{elt}}}(-\mathbf{x},\nu)\,u(\mathbf{x},\nu)\,d\mathbf{x}.$$

If $h_{\mathbf{x}_{\mathrm{elt}}}(\mathbf{x},\nu)=\delta(\mathbf{x}+\mathbf{x}_{\mathrm{elt}})$, then this element output integrates to $u(\mathbf{x}_{\mathrm{elt}},\nu)$, the field sampled at $\mathbf{x}=\mathbf{x}_{\mathrm{elt}}$. Therefore, the Fourier transform $H_{\mathbf{x}_{\mathrm{elt}}}(\ell,\nu)=\mathrm{e}^{\mathrm{j}2\pi\ell\cdot\mathbf{x}_{\mathrm{elt}}}$ is just the pattern of an isotropic element located at $\mathbf{x}=\mathbf{x}_{\mathrm{elt}}$.

By a similar argument, $H_{\mathbf{x}_{\mathrm{elt}}}(\ell,\nu)=G_{\mathbf{x}_{\mathrm{elt}}}(\ell,\nu)\,\mathrm{e}^{\mathrm{j}2\pi\ell\cdot\mathbf{x}_{\mathrm{elt}}}$ is just the pattern of an element that scales the field by complex gain $G_{\mathbf{x}_{\mathrm{elt}}}(\ell,\nu)$ before sampling at $\mathbf{x}=\mathbf{x}_{\mathrm{elt}}$. Substitution into (3) yields a convenient alternative form in which elements physically identical except for position are represented by simply removing the indexing subscript from $G_{\mathbf{x}_{\mathrm{elt}}}(\ell,\nu)$:

$$Y(f)=\int_{\mathbb{R}^3}\int_{\mathbb{R}} A(\ell;f,\nu)\,U(\ell,\nu)\,d\nu\,d\ell, \tag{4}$$

$$A(\ell;f,\nu)=\sum_{x_{\mathrm{elt}}\in\mathcal{X}} D_{\mathbf{x}_{\mathrm{elt}}}(f,\nu)\,G_{\mathbf{x}_{\mathrm{elt}}}(\ell,\nu)\,\mathrm{e}^{\mathrm{j}2\pi\ell\cdot\mathbf{x}_{\mathrm{elt}}}\,. \tag{5}$$

Array pattern $A(\ell;f,\nu)$ has been defined for convenience. When the elements are identical, by convention we leave element pattern $G(\ell,\nu)$ outside of $A(\ell;f,\nu)$, now called the array factor:

$$Y(f)=\int_{\mathbb{R}^3}\int_{\mathbb{R}} A(\ell;f,\nu)\,G(\ell,\nu)\,U(\ell,\nu)\,d\nu\,d\ell, \tag{6}$$

$$A(\ell;f,\nu)=\sum_{x_{\mathrm{elt}}\in\mathcal{X}} D_{\mathbf{x}_{\mathrm{elt}}}(f,\nu)\,\mathrm{e}^{\mathrm{j}2\pi\ell\cdot\mathbf{x}_{\mathrm{elt}}}\,. \tag{7}$$

It is intuitively satisfying that system outputs (4) and (6) now comprise sums of element outputs first processed through individual linear, time-varying systems. It is straightforward to confirm that if $D_{\mathbf{x}_{\mathrm{elt}}}(f,\nu)$ is time invariant, so that it can be written $D_{\mathbf{x}_{\mathrm{elt}}}(f)\,\delta(\nu-f)$, (5) and (7) become the familiar LTI wideband array pattern and array factor. They further specialize to the usual narrowband-array formulations if the LTI frequency response is simply a phase shift constant over the band of interest.

Now let us examine a less-familiar specialization.

3. Periodic Signal Processing

Periodic signal processing will yield impulse responses time invariant with respect to some $T\mathbb{Z}$ (A.2) and therefore the structure

$$D_{\mathbf{x}_{\text{elt}}}(f,\nu) = \sum_{k\in\mathbb{Z}} D_{k,\,\mathbf{x}_{\text{elt}}}(f-k/T)\,\delta(\nu-(f-k/T)),$$

where $D_{k,\,\mathbf{x}_{\text{elt}}}$ is defined here to include the k/T offset in its argument for convenience. Formulation (4) now becomes

$$\begin{aligned} Y(f) &= \sum_{k\in\mathbb{Z}} Y_k(f-k/T), \qquad (8)\\ Y_k(f) &= \int_{\mathbb{R}^3} A_k(\ell,f)\,U(\ell,f)\,d\ell,\\ A_k(\ell,f) &= \sum_{x_{\text{elt}}\in\mathcal{X}} D_{k,\,\mathbf{x}_{\text{elt}}}(f)\,G_{\mathbf{x}_{\text{elt}}}(\ell,f)\,\mathrm{e}^{\mathrm{j}2\pi\ell\cdot\mathbf{x}_{\text{elt}}}\,. \end{aligned}$$

Effectively, separate LTI array patterns and LTI filters generate, quite conventionally, components that are summed after shifting each by a different multiple of fundamental frequency $1/T$. Identical element patterns modify this only slightly:

$$\begin{aligned} Y(f) &= \sum_{k\in\mathbb{Z}} Y_k(f-k/T), \qquad (9)\\ Y_k(f) &= \int_{\mathbb{R}^3} A_k(\ell;f)\,G(\ell,f)\,U(\ell,f)\,d\ell,\\ A_k(\ell;f) &= \sum_{x_{\text{elt}}\in\mathcal{X}} D_{k,\,\mathbf{x}_{\text{elt}}}(f)\,\mathrm{e}^{\mathrm{j}2\pi\ell\cdot\mathbf{x}_{\text{elt}}}\,. \end{aligned}$$

4. Summary and Conclusions

With input taken as $U(\ell,\nu)$, the Fourier spectrum of incident plane waves as a function of spatial and temporal frequencies ℓ and ν (related by c), this paper develops the general input-output behavior of a linear, time-varying receive system based on an array of physically static elements of arbitrary geometry. It is shown here that every such system behaves according to (4), which describes a sum of element outputs first processed through individual linear, time-varying systems. Identical element patterns specialize this result to (6), periodic signal processing specializes it to (8), and these two specializations together result in (9).

All time-varying array systems known to this author, existing or proposed, operate periodically. And any system that must cope with a multi-signal input spectrum, especially if arbitrarily large extraneous signals are possible, must be (at least nominally) linear. So formulation (8) is fully general in a practical sense. It shows that an arriving input signal may be changed arbitrarily in amplitude and phase as it passes through the system but that the only possible frequency shifts are by integer multiples of the reciprocal of the period of the variation in the signal processing. This is quite limiting. It means, for example, that creating a frequency shift that is a smooth function of signal direction of arrival is fundamentally impossible with this class of system.

It remains to be determined what systems might be useful within the framework determined here. But now we know where to begin searching. We begin with (9).

Appendix A
Time-Varying Linear Systems

Few electrical engineers are familiar with effective ways of dealing mathematically with time-varying linear systems, so a brief tutorial is developed here. The aim is to connect intuitively with the impulse-response and frequency-response characterizations of LTI systems while both raising the mathematical level enough to make it interesting to (say) an engineering professor and omitting details to respect limitations on her time. (The corresponding discrete-time theory is developed, somewhat more thoroughly, by Akkarakaran and Vaidyanathan [2].)

A1 One-Dimensional Systems

The various forms of the celebrated Riesz representation theorem of mathematical analysis state with suitable mathematical hairsplitting that a linear system operating on complex-valued input $v(t)$ to produce complex-valued output $y(t)$ can be represented by

$$y(t) = \int_{\mathbb{R}} v(\tau)\,d\rho_t(\tau).$$

with respect to a t-dependent complex measure ρ_t that characterizes the system. If ρ_t is absolutely continuous with respect to Lebesgue measure on $\mathbb{R}$, or if we allow impulsive components in impulse responses and accept their limitations for representing measure components singular with respect to Lebesgue measure, some form—more hairsplitting—of the Radon-Nikodym theorem allows this measure to be represented in terms of some complex impulse response function $g(t,-\tau)$, where the minus sign is simply a convenient convention, in a time-varying convolution integral:

$$\begin{aligned} y(t) &= \int v(\tau)\,g(t,-\tau)\,d\tau\\ &= \int v(\tau)\,\left(g^*(t,-\tau)\right)^*\,d\tau. \end{aligned}$$

The notation $g(t,-\tau)$ was chosen because it is the system's response to $\delta(t-\tau)$. The double conjugation reveals an inner product in τ, so assuming L_2 (finite energy) functions of τ, Parseval's relation (the Fourier transform as a Hilbert-space isomorphism) yields

$$y(t) = \int \left(G_t^*(\nu)\right)^*\,V(\nu)\,d\nu$$

where $V(\nu)$ and $G_t(\nu)$ are the Fourier transforms on τ of $v(\tau)$ and $g(t,\tau)$. Transforming on t with f as the new frequency variable,

$$Y(f) = \int G(f,\nu)\,V(\nu)\,d\nu \qquad \text{(A.10)}$$

where $G(f,\nu)$ is the 2D Fourier transform of $g(t,\tau)$ given by

$$G(f,\nu) = \iint g(t,\tau)\, \mathrm{e}^{-\mathrm{j}2\pi(ft+\nu\tau)}\; dt\, d\tau.$$

Equation (A.10) is just $Y = G \star V$ using a noncommutative convolution operator $\star$, and it will be our primary means of representing time-varying linear systems in one dimension. Frequency response $G(f,\nu)$ is the complex gain coupling input component at frequency ν to the output component at frequency f.

A2 Time Invariance of Systems

Adopting (appropriately modified) terminology from [3], let us say a linear system is *invariant* with respect to set $T\mathbb{Z}$ of times if shifting the input by T simply shifts the output by T as well. Using impulse-response convolution, this becomes

$$\begin{aligned}\int g(t,-\tau)\, v(\tau-T)\, d\tau &= \int g(t-T,-\tau)\, v(\tau)\, d\tau \\ &= \int g(t-T,-\tau+T)\, v(\tau-T)\, d\tau,\end{aligned}$$

using a change of variable. The first equality holds for all inputs $v(t)$ if and only if $g(t,\tau) = g(t-T,\tau+T)$ for all t and (almost) all τ. Fourier transformation yields $G(f,\nu) = G(f,\nu)\, \mathrm{e}^{-\mathrm{j}2\pi(f-\nu)T}$, which implies that $G(f,\nu) = 0$ unless $(f-\nu)T \in \mathbb{Z}$. Therefore, time invariance with respect to $T\mathbb{Z}$ implies the structure

$$G(f,\nu) = \sum_{k\in\mathbb{Z}} G_k(f)\, \delta(\nu - (f - k/T)), \qquad \text{(A.11)}$$

and (A.10) then becomes

$$Y(f) = \sum_{k\in\mathbb{Z}} G_k(f)\, V(f - k/T),$$

revealing that such a system simply creates spectral replicas at $1/T$ intervals and filters them separately.

A3 Elementary Examples and Properties

Example: Conventional LTI filter. A system invariant with respect to $T\mathbb{Z}$ for all $T \in \mathbb{R}$ is said simply to be *time invariant*. This familiar LTI system has $G(f,\nu) = G_0(f)\, \delta(\nu - f)$, the general form (A.11) with just one nonzero term. Inverse Fourier transforming,

$$\begin{aligned} g(t,\tau) &= \iint G_0(f)\, \delta(\nu - f)\, \mathrm{e}^{\mathrm{j}2\pi(ft+\nu\tau)}\, df\, d\nu \\ &= \int G_0(f)\, \mathrm{e}^{\mathrm{j}2\pi f(t+\tau)}\, df \;=\; g_0(t+\tau). \end{aligned}$$

The dependence of impulse response $g(t,-\tau) = g_0(t-\tau)$ on t and τ is only through dependence on $t-\tau$ as expected. Generally the subscripts on $g_0(t)$ and $G_0(f)$ are dropped and these are termed the system impulse response and frequency response respectively. Now (A.10) becomes

$$Y(f) = \int G(f)\, \delta(\nu - f)\, V(\nu)\, d\nu = G(f)\, V(f). \qquad \text{(A.12)}$$

□

Example: Multiplication by a fixed waveform. Multiplication by a fixed $w(t)$ has an associated frequency response $G(f,\nu)$ that is immediately recognizable as $W(f-\nu)$ through comparison of (A.10) with the familiar Fourier property

$$\int v(t)\, w(t)\, \mathrm{e}^{-\mathrm{j}2\pi ft}\, dt = \int V(\nu)\, W(f-\nu)\, d\nu. \qquad \text{(A.13)}$$

□

Example: A cascade. If $Z = H \star Y$ and $Y = G \star V$, then

$$\begin{aligned} Z(f) &= \int H(f,\nu)\, \overbrace{\int G(\nu,\mu)\, V(\mu)\, d\mu}^{(G\star V)(\nu)}\, d\nu \\ &= \int \underbrace{\int H(f,\nu)\, G(\nu,\mu)\, d\nu}_{\text{call it } (H\star G)(f,\mu)}\, V(\mu)\, d\mu. \end{aligned}$$

So in this notation, $H\star(G\star V) = (H\star G)\star V$, making our new convolution operator associative. The structural similarity of this frequency-domain convolution integral (A.10) to a matrix multiplication is certainly intuitively satisfying. □

Example: Filter, then multiply. IQ demodulation is an example comprising filtering, to isolate the positive-frequency portion of the bandpass input spectrum, followed by multiplication by a complex exponential to shift the carrier frequency to the origin. LTI filtering with impulse response $h(t-\tau)$ followed by multiplication by $w(t)$ results in overall frequency response

$$\begin{aligned} G(f,\nu) &= \int W(f-\mu)\, H(\mu)\, \delta(\nu-\mu)\, d\mu \\ &= W(f-\nu)\, H(\nu). \end{aligned}$$

□

Example: Multiply, then filter. The same IQ demodulation can be accomplished with frequency shifting followed by lowpass filtering. When multiplication precedes filtering,

$$\begin{aligned} G(f,\nu) &= \int H(f)\, \delta(\mu - f)\, W(\mu-\nu)\; d\mu \\ &= H(f)\, W(f-\nu). \end{aligned}$$

□

The last two examples are completely intuitive if $G(f,\nu)$ is seen as a gain coupling input frequency ν to output frequency f.

A4 Multiple Dimensions

The extension of the above to multiple dimensions is simple. Independent variables become multidimensional, and integration is over a multidimensional space. For example, a linear system with input $u(\mathbf{x},t)$ and output $y(\mathbf{x},t)$, each a function of the three spatial dimensions of $\mathbf{x}$ and the one temporal dimension of t, is characterized by impulse response $g(\mathbf{x}, -\mathbf{y}; t, -\tau)$ and a convolution integral over vector $\mathbf{y}$ and scalar τ:

$$y(\mathbf{x}, t) = \int_{\mathbb{R}} \int_{\mathbb{R}^3} u(\mathbf{y}, \tau)\, g(\mathbf{x}, -\mathbf{y}; t, -\tau)\, d\mathbf{y}\, d\tau.$$

The associated frequency-response convolution integral (with spatial frequencies ℓ and η and temporal frequency f and ν) is

$$Y(\ell, f) = \int_{\mathbb{R}} \int_{\mathbb{R}^3} G(\ell, \eta; f, \nu)\, U(\eta, \nu)\, d\eta\, d\nu.$$

Here the ";" is adopted to separate variables corresponding to different dimensions. All of the Fourier transforms are multidimensional in exactly the way expected.

Appendix B
A 4D Incident Electromagnetic Field

Consider a single polarization component of the electric or magnetic field, normalized to remove units. A wide range of such scalar functions $u(\mathbf{x}, t)$ of 3D spatial position $\mathbf{x}$ and time t can be represented in terms of their 4D Fourier transforms $U(\ell, f)$, where ℓ is a 3D spatial frequency and f is temporal frequency, as

$$u(\mathbf{x}, t) = \int_{\mathbb{R}^3} \int_{\mathbb{R}} U(\ell, f)\, \mathrm{e}^{\mathrm{j}2\pi(\ell\cdot\mathbf{x}+ft)}\, df\, d\ell. \qquad \text{(B.14)}$$

If $u(\mathbf{x}, t)$ is real and there is no electrostatic or $f = 0$ component, symmetry condition $U(\ell, f) = U^*(-\ell, -f)$ permits this to be written

$$u(\mathbf{x}, t) = \mathrm{Re}\left\{ \int_{\mathbb{R}^3} \int_0^\infty 2U(-\mathbf{k}, f)\, \mathrm{e}^{\mathrm{j}2\pi(ft-\mathbf{k}\cdot\mathbf{x})}\, df\, d\mathbf{k} \right\}$$

where convenient change of variable $\mathbf{k} = -\ell$ points ahead toward a conventional representation of propagating plane waves.

To obtain just such a representation, scale by reference constant E_{ref} with (say) electric-field units and suppose that for $f > 0$ the function $U(\ell, f)$ is such that $E_{\mathrm{ref}}\, 2U(-\mathbf{k}, f) = E(\mathbf{k})\, \delta(f - c\|\mathbf{k}\|)$, where constant c has velocity dimensions, so that $U(\ell, f)$ contains only components for which the Helmholtz equation $f = c\,\|\mathbf{k}\|$ holds. This reduces the effective dimensionality of the integrand by one, which is made explicit by integrating out f:

$$E_{\mathrm{ref}}\, u(\mathbf{x}, t) = \mathrm{Re}\left\{ \int_{\mathbb{R}^3} E(\mathbf{k})\, \mathrm{e}^{\mathrm{j}2\pi(ft-\mathbf{k}\cdot\mathbf{x})}\, d\mathbf{k} \right\}.$$

The f has been left in the exponent as a notational convenience only, with the understanding that here it is shorthand for $c\,\|\mathbf{k}\|$.

The integrand is the complex electric field associated with a plane-wave component at wavenumber (spatial frequency) $\mathbf{k}$, temporal frequency $f = c\,\|\mathbf{k}\|$, and complex amplitude $E(\mathbf{k})$. Factor $\delta(f - c\|\mathbf{k}\|)$ in 4D spectrum $U(\ell, f)$ restricts wave propagation to velocity c, even though the 4D spectral integral (B.14) can represent waves propagating at all velocities simultaneously.

Realizing then that specializations of $U(\ell, f)$ are involved, we represent (a single polarization component of) the electromagnetic field incident on an array with (B.14), because its straightforward Fourier structure makes all the usual Fourier properties immediately available. Looking in direction ℓ, we see an arriving plane wave with wavenumber $\mathbf{k} = -\ell$ and of magnitude f/c, so we interpret ℓ as a *look-direction* unit vector that has been scaled by inverse wavelength. We typically omit conversion constant E_{ref} in derivations, however, understanding that it can be put back when needed to deal with physical reality.

References

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